Why does the Gauss-Jordan method succeed in finding the inverse A^{-1} of a square matrix A?

To understand this, we need to understand row operations.

Row operation I

e.g.

$$\underset{A}{\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 7 & 1 & -2 \end{pmatrix}} \xrightarrow{R_2 \mapsto R_2 - 2R_3} \underset{B}{\begin{pmatrix} 1 & 3 & 5 \\ -12 & -6 & 10 \\ 7 & 1 & -2 \end{pmatrix}}$$

$$\underset{E}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}} \underset{A}{\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 7 & 1 & -2 \end{pmatrix}} = \underset{B}{\begin{pmatrix} 1 & 3 & 5 \\ -12 & -6 & 10 \\ 7 & 1 & -2 \end{pmatrix}}$$

Row operation II

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 7 & 1 & -2 \end{pmatrix} \xrightarrow{R_3 \to 3R_3} \begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 21 & 3 & -6 \end{pmatrix}$$

$A \qquad\qquad B$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 7 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 21 & 3 & -6 \end{pmatrix}$$

$E \qquad A \qquad B$

Row operation III

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 7 & 1 & -2 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 3 & 5 \\ 7 & 1 & -2 \\ 2 & -4 & 6 \end{pmatrix}$$

$A \qquad\qquad B$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 7 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 \\ 7 & 1 & -2 \\ 2 & -4 & 6 \end{pmatrix}$$

$E \qquad A \qquad B$

Let's look again at Gauss-Jordan method.

If

$$\left(A \;\vdots\; I \right) \xrightarrow[\text{operations}]{\text{row}} \left(I \;\vdots\; B \right)$$

then there are matrices $E_1, E_2, E_3, \ldots, E_k$ such that

$$(E_k \cdots E_3 E_2 E_1) A = I$$

So

$$(E_k \cdots E_3 E_2 E_1) A A^{-1} = I A^{-1}$$

and

$$(E_k \cdots E_3 E_2 E_1) I = A^{-1}$$

This (kind of) proves why the Gauss-Jordan method works.

Example A factory requires energy, steel and labour to manufacture machines of type A, B, C.

Resource	A	B	C	weekly amount
energy	2 Mwh	3 Mwh	2 Mwh	100 Mwh
steel	1 tonne	1 tonne	4 tonne	70 tonnes
labour	20 hrs	10 hrs	10 hrs	500 hrs

What production figures ensure that all resources are used?

Sol^n Let's suppose we manufacture

x units of machine A

y " " " B

z " " " C

If all resources are used, then

$$\left.\begin{aligned} 2x + 3y + 2z &= 100 \\ 1x + 1y + 4z &= 70 \\ 20x + 10y + 10z &= 500 \end{aligned}\right\} (*)$$

Sol^n

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 4 \\ 20 & 10 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 100 \\ 70 \\ 500 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 4 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 100 \\ 70 \\ 50 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 4 \\ 2 & 3 & 2 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 70 \\ 100 \\ 50 \end{pmatrix}$$

$R_2 \longmapsto R_2 - 2R_1$

$R_3 \longmapsto R_3 - 2R_1$

$$\begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & -1 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 70 \\ -40 \\ -90 \end{pmatrix}$$

$R_3 \longrightarrow R_3 + R_2$

$$\begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & 0 & -13 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 70 \\ -40 \\ -130 \end{pmatrix}$$

$$x + y + 4z = 70$$

$$y - 6z = -40$$

$$-13z = -130$$

$z = 10$

$y = 20$

$x = 10$

$y = -40 + 60$

$x = 70 - 20 - 40$