Last lecture:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\det(A) = ad - bc$

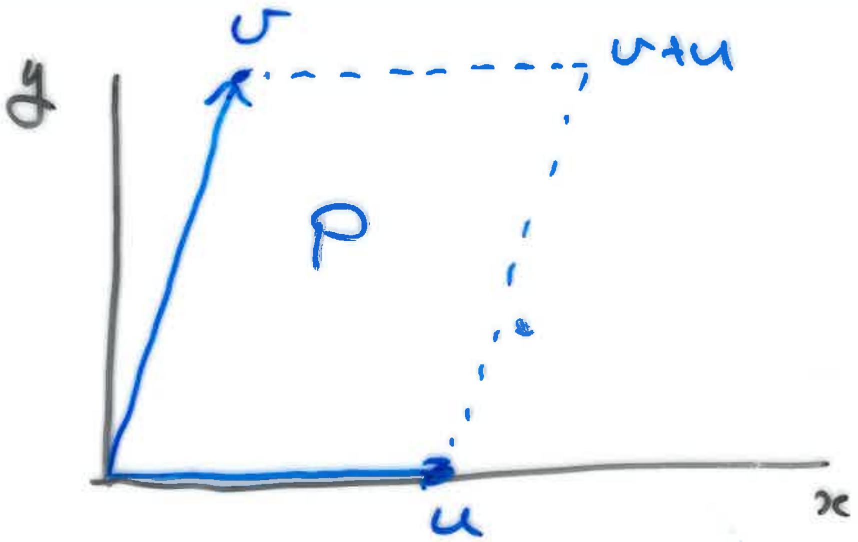
We'll also write $|A| = ad - bc$.

<u>Example</u>

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 5 \end{pmatrix}$$

$\det(A) = |A| = 3.5 - 0.1 = 15$

Consider $u = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ $v = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$



Area of $P = 3 \times 5 = 15$

Note: In this example

$\det(A) =$ area of P.

The calculations of the area of P are easy because u lies on the x-axis.

Example

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 \\ 3 & -4 \end{pmatrix}$$

$$\det(A) = 1 \cdot 4 - 3 \cdot 2 = -2$$

$$\det(B) = 2(-4) - (3)(-1) = -5$$

$$\det(AB) = \det \begin{pmatrix} 8 & -9 \\ 18 & -19 \end{pmatrix}$$

$$= 8(-19) - (18)(-9)$$

$$= 10$$

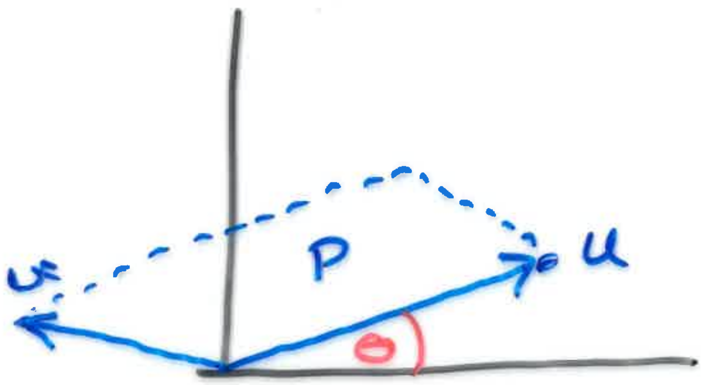
Note: $\det(AB) = \det(A) \cdot \det(B)$

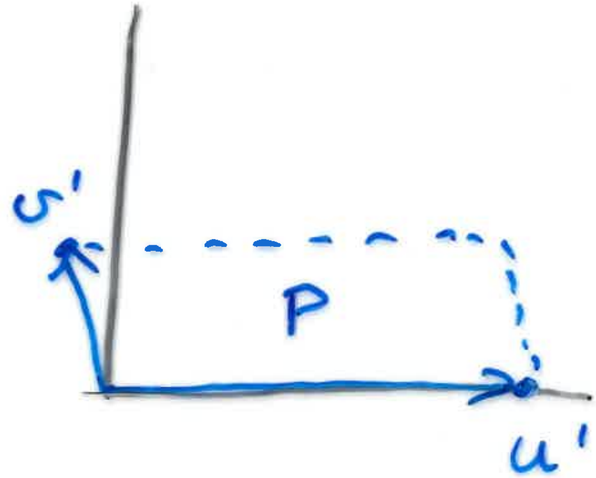
This example easily generalizes to

Theorem For any 2×2 matrices A, B we have

$$|AB| = |A| . |B|$$

Towards a proof that areas of parallelograms are captured by determinants.





$$u = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} u'$$

$$v = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} v'$$

$$\det\begin{pmatrix} \vert & \vert \\ u & v \\ \vert & \vert \end{pmatrix}$$

$$= \det\left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \vert & \vert \\ u' & v' \\ \vert & \vert \end{pmatrix} \right)$$

$$= \det\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \det\begin{pmatrix} \vert & \vert \\ u' & v' \\ \vert & \vert \end{pmatrix}$$

$$= (\cos^2\theta + \sin^2\theta)\, \det\begin{pmatrix} | & | \\ u' & v' \\ | & | \end{pmatrix}$$

$$= \det\begin{pmatrix} | & | \\ u' & v' \\ | & | \end{pmatrix}$$

$= \pm$ area of P.

Eigenvalues & Eigenvectors

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix of real numbers.

Definition A non-zero vector

$$v = \begin{pmatrix} x \\ y \end{pmatrix}$$

is an eigenvector for A if there exists some real number λ such that

$$A v = \lambda v.$$

We call λ the eigenvalue of A corresponding to v.

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Consider

$$v = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

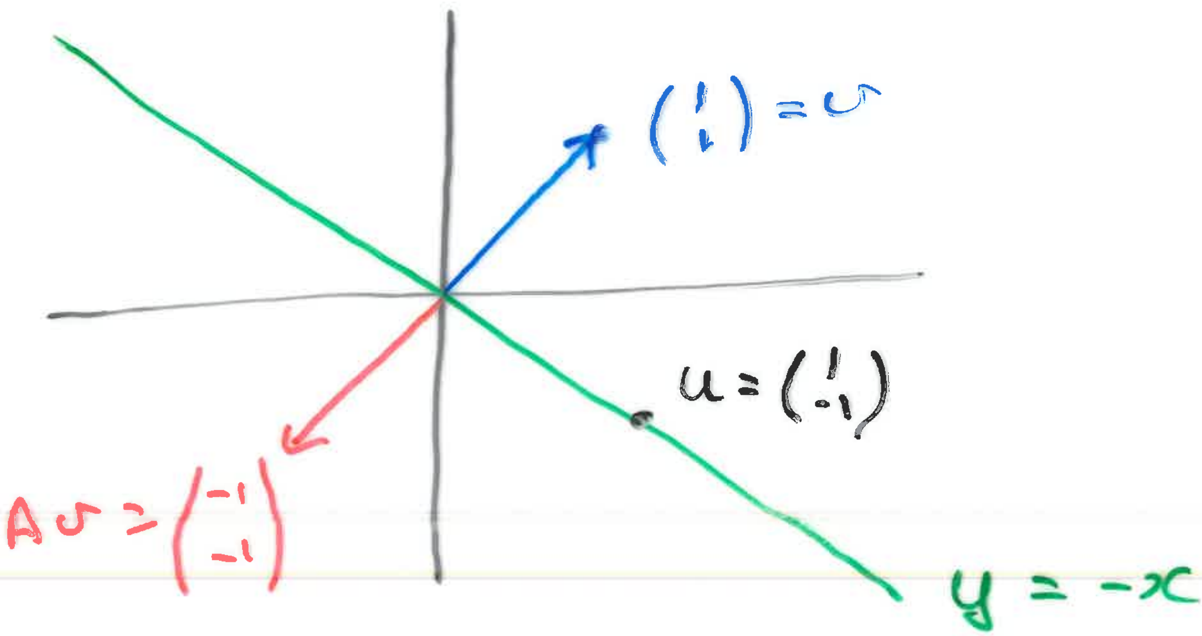
Then

$$\underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 4 \\ 4 \end{pmatrix}}_{v} = \begin{pmatrix} 12 \\ 12 \end{pmatrix} = 3 \underbrace{\begin{pmatrix} 4 \\ 4 \end{pmatrix}}_{v}$$

Thus v is an eigenvector for A with corresponding eigenvalue $\lambda = 3$.

Example

Let A be the Matrix of reflection in the line $y = -x$.



So $Av = -v$ for $v = \begin{pmatrix}1\\1\end{pmatrix}$

Hence v is an eigenvector of A with corresponding eigenvalue $\lambda = -1$.

Also $Au = u$ for $u = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Hence $u = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector for A with eigenvalue $\lambda = 1$.

<u>Example</u> Give me a matrix A that has <u>no</u> eigenvectors.

<u>Answer</u> Let A be the matrix of rotation about the origin through an angle θ with $\theta \neq 0, \pi$. Then "clearly" A has no eigenvector.