Last lecture

The characteristic polynomial $P_A(\lambda)$ of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

is

$$P_A(\lambda) = \det(A - \lambda I)$$

$$= \det\left(\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$$

$$= \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)(1-\lambda) - 2.2$$

$$= \lambda^2 - 2\lambda + 1 - 4$$

$$= \lambda^2 - 2\lambda - 3$$

$$P_A(\lambda) = \lambda^2 - 2\lambda - 3\lambda^0$$

$$P_A(2) = 2^2 - 2.2 - 3 = -3$$

$$P_A(A) = A^2 - 2A - 3I$$

$$= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Cayley-Hamilton Theorem

For any 2×2 matrix A

we have

$$P_A(A) = 0I$$

Calculating Eigenvalues & eigenvectors

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix of real numbers.

A non-zero vector $v = \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is said to be an eigenvector of A if

$$Av = \lambda v$$

for some number λ. We say that λ is the eigenvalue of A corresponding to v.

To calculate eigenvalues/vectors we need the following result.

Proposition Let A be a 2×2 matrix of real numbers, and let $v = \begin{pmatrix} x \\ y \end{pmatrix}$ be some non-zero vector, and suppose that

$$Av = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (*)$$

Then $\det(A) = 0$.

Proof If A^{-1} existed then from $(*)$ we get

$$A^{-1}Av = A\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and so

$$v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We conclude that, under the hypothesis of the Proposition, A^{-1} does <u>not</u> exist.

Recall that

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Since A^{-1} does not exist, and since $\operatorname{adj}(A)$ always exists, we get that

$$\det(A) = 0. \qquad \text{QED}$$

How can we find eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} ?$$

Suppose $v = \begin{pmatrix} x \\ y \end{pmatrix}$ is some eigenvector of A. Then

$$Av = \lambda v$$

for some number λ.

$$Av = \lambda v$$

$$Av - \lambda v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$Av - \lambda I v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(A - \lambda I) v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So $\boxed{\det(A - \lambda I) = 0}$

by the proposition above and the fact that eigenvectors are non-zero.

So

$$\det(A - \lambda I) = 0$$

$$P_A(\lambda) = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda + 1)(\lambda - 3) = 0$$

Hence the eigenvalues of A are $\lambda = -1$ and $\lambda = 3$.

Now let's find eigenvectors.

<u>Case $\lambda = -1$</u>

Need $Av = \lambda v$

or $Av = -v$

or

$$Av + v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$(A + I)v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

e.g. $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of A.

Case $\lambda = 3$

Need $Av = \lambda v$

$Av = 3v$

$$(A - 3I)v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For instance

$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector

corresponding to $\lambda = 3$.