Rabbit Populations

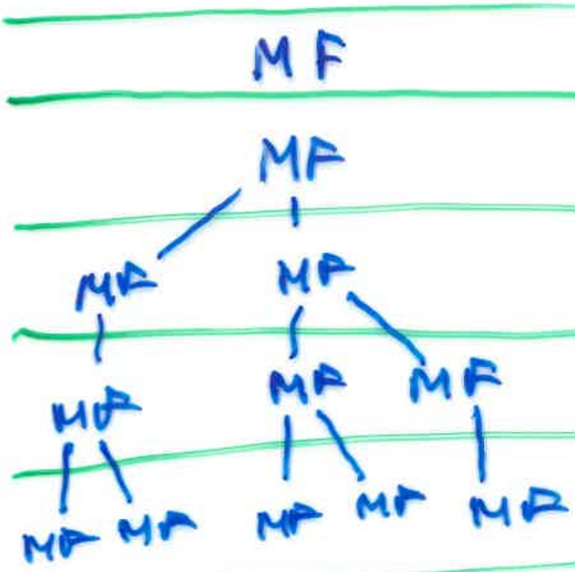
- One newly born male rabbit and one newly born female rabbit are placed in a field.

- Rabbits can mate at the age of one month, and one month later the female produces one male/female pair of kittens

- Rabbits never die

- How fast does the rabbit population grow?

- How many rabbits will there be after 100 month?



Month	number of pairs
0	1
1	1
2	2
3	3
4	5
5	8
6	13
7	21
8	34
9	55
10	84
11	144
12	233

Let F_n = number of pairs of rabbits after n months

$$\boxed{F_{n+2} = F_{n+1} + F_n}$$

Q? what value is F_{100}?

Let's look at

F_n / F_{n-1}

to see how quickly the rabbit population grows

$\frac{1}{1}$ $\frac{2}{1}$ $\frac{3}{2}$ $\frac{5}{3}$ $\frac{8}{5}$ .... $\frac{55}{34}$

1.618

$\frac{84}{55}$ ... $\frac{233}{144}$

1.618

sunflowers

belly - buttons

windows

Maybe the Sequence

$$\frac{F_n}{F_{n-1}}$$

Converges as $n \to \infty$?

Maybe

$$\lim_{n \to \infty} \frac{F_n}{F_{n-1}} = \phi .$$

How would we calculate

ϕ ?

If $\lim_{n \to \infty} \frac{F_n}{F_{n-1}} = \phi$ then,

for very large n,

$$\frac{F_n}{F_{n-1}} \simeq \phi .$$

or

$$\boxed{F_n \simeq \phi F_{n-1}}$$

or

$$\phi \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} \simeq \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$

So ϕ should be an eigenvalue of A.

$$\boxed{\begin{aligned} F_n &= F_{n-1} + F_{n-2} \\ F_{n-1} &= F_{n-1} \end{aligned}}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^3 \begin{pmatrix} F_{n-3} \\ F_{n-4} \end{pmatrix}$$

$$\vdots$$

$$\boxed{\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}}$$

To find the eigenvalues of A:

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0$$

$$\det\begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(-\lambda) - 1.1 = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{+1 \pm \sqrt{1+4}}{2}$$

Eigenvalues of A are

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \simeq 1.618\ldots, \quad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

The number $\boxed{\phi = \frac{1+\sqrt{5}}{2}}$ is called the Golden ratio.