Rabbits

$F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5, \ldots$

where

$$F_n = F_{n-1} + F_{n-2}.$$

Aim: Find an explicit formula for F_n in terms of n but <u>not</u> involving F_{n-1}, F_{n-2}.

Theorem If a 2×2 matrix A has eigenvalues λ_1, λ_2 with corresponding eigenvectors v_1, v_2, and if the matrix

$$T = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$$

is invertible, then

$$T^{-1} A T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Proof

$$T^{-1}AT = T^{-1}A\begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$$

$$= T^{-1}\begin{pmatrix} | & | \\ \lambda_1 v_1 & \lambda_2 v_2 \\ | & | \end{pmatrix}$$

$$= T^{-1}\begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$= T^{-1}T\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

QED

Example Consider

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

We've seen that the eigenvalues of A are:

$$\lambda_1 = \phi = \frac{1+\sqrt{5}}{2}$$

$$\lambda_2 = \bar{\phi} = \frac{1-\sqrt{5}}{2}$$

Note $\phi \bar{\phi} = -1$

Let's find corresponding eigenvectors. Need to solve

$$(A - \lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for $\lambda = \phi, \bar{\phi}$.

Consider $\lambda = \phi$.

$$\begin{pmatrix} 1-\phi & 1 \\ 1 & -\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

one solution

$$\begin{pmatrix} 1-\phi & 1 \\ 1 & -\phi \end{pmatrix} \underbrace{\begin{pmatrix} \phi \\ 1 \end{pmatrix}}_{\text{eigenvector for } \lambda = \phi} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Consider $\lambda = \bar{\phi}$

$$\begin{pmatrix} 1-\bar{\phi} & 1 \\ 1 & -\bar{\phi} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

one solution

$$\begin{pmatrix} 1-\bar{\phi} & 1 \\ 1 & -\bar{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ -\phi \end{pmatrix}}_{\text{eigenvector for } \lambda = \bar{\phi}.} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A^n = (TDT^{-1})^n$$

$$= (TD\cancel{T^{-1}})(\cancel{T}D\cancel{T^{-1}})(\cancel{T}D\cancel{T^{-1}}) \cdots\cdots (\cancel{T}DT^{-1})$$

$$= T D^n T^{-1}$$

$$= T \begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix}^n T^{-1}$$

$$= T \begin{pmatrix} \phi^n & 0 \\ 0 & \bar{\phi}^n \end{pmatrix} T^{-1}$$

By the above theorem

$$\boxed{T^{-1} A T = \begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix}}$$

with $T = \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Recall from last lecture

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = A^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

we have

$$\boxed{A = T \underbrace{\begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix}}_{D} T^{-1}}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = A^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = T D^{n-1} T^{-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix} \begin{pmatrix} \phi^{n-1} & 0 \\ 0 & \bar{\phi}^{n-1} \end{pmatrix} \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Exercise

$$F_n = \frac{1}{\sqrt{5}} \phi^n - \frac{1}{\sqrt{5}} \bar{\phi}^n$$