Clock arithmetic is very much like the arithmetic in school. In particular:

1) $a+b = b+a$ (commutative)

1') $ab = ba$

2) $(a+b)+c = a+(b+c)$ (associative)

2') $(ab)c = a(bc)$

3) $a(b+c) = ab+ac$ (distributive)

We'll now study an arithmetic where property (1') fails.

Matrix Arithmetic

A matrix is an array of numbers arranged neatly in rows and columns. Each row has the same length, and each column has the same length.

Example

$$\begin{pmatrix} 1 & 2 & 5 \\ -2 & 3 & 10 \end{pmatrix}$$ 2×3 matrix

$$\begin{pmatrix} \frac{1}{2} & -\sqrt{2} \\ 3 & 7 \end{pmatrix}$$ 2×2 matrix

$$\begin{pmatrix} 1 & 2 & 3 & -4 & 5 \end{pmatrix}$$ 1×5 matrix

also a "row vector"

$$\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$ 3×1 matrix

also a "column vector"

Matrix Addition

Two $m \times n$ matrices A, B are added by adding corresponding entries.

$$\underset{A}{\begin{pmatrix} 17 & 22 & 42 \\ 6 & 18 & 5 \end{pmatrix}} + \underset{B}{\begin{pmatrix} 1 & 2 & 3 \\ -6 & 1 & 5 \end{pmatrix}} = \underset{A+B}{\begin{pmatrix} 18 & 24 & 45 \\ 0 & 19 & 10 \end{pmatrix}}$$

$$\underset{A}{\begin{pmatrix} -2 & 3 \\ 1 & 4 \end{pmatrix}} + \underset{A}{\begin{pmatrix} 7 & 6 \\ 5 & 4 \end{pmatrix}} = \begin{pmatrix} 5 & 9 \\ 6 & 8 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}} + \underset{B}{\begin{pmatrix} 1 & -2 & 3 & -4 \end{pmatrix}} = \underset{A+B}{\begin{pmatrix} 2 & 0 & 6 & 0 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix}$$

Can not be added.

Given a matrix A we write $-A$ to denote the matrix got from A by placing a "$-$" in front of each entry.

$$A = \begin{pmatrix} 1 & -2 \\ 3 & -5 \end{pmatrix}$$

$$-A = \begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix}$$

Note that

$A + (-A) =$ matrix with all entries equal to 0 (and of the same dimensions as A).

$$\begin{pmatrix} 1 & -2 \\ 3 & -5 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

A matrix whose entries are all zero is called a zero matrix, and often denoted by 0.

Thus, for any matrix A we have

$$A + (-A) = 0$$

Multiplication of a row vector by a column vector

Let

$R = (a_1, a_2, \dots, a_n)$

be a row vector of length n.

Let

$$C = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

be a column vector of length n.

We define

$$R.C = (a_1 \ a_2 \cdots a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

<u>Example</u> $R = (-2 \ 3 \ 7)$

$$C = \begin{pmatrix} 8 \\ 5 \\ 9 \end{pmatrix}$$

$$R.C = (-2)8 + 3.5 + 7.9$$

$$= -16 + 15 + 63$$

$$= 62$$

Matrix Multiplication

A matrix

$$A = \begin{array}{c} \begin{pmatrix} 1 - 2 - 3 \\ 4 - 5 - 6 \end{pmatrix} \end{array} \begin{array}{c} R_1 \\ R_2 \end{array}$$

can be regarded as a collection of rows.

A matrix

$$B = \begin{array}{ccc} c_1 & c_2 & c_3 \\ \hline -1 & 2 & 1 \\ | & | & | \\ 1 & 3 & 3 \\ | & | & | \\ 0 & 2 & 4 \end{array}$$

can be regarded as a collection of columns.

Let A be an $m \times \underline{n}$ matrix.

Let B be an $\underline{n} \times p$ matrix.

We define the product

AB to be

$$\underbrace{\begin{pmatrix} — & R_1 & — \\ — & R_2 & — \\ & \vdots & \\ — & R_m & — \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} | & | & & | \\ C_1 & C_2 & \cdots & C_p \\ | & | & & \vdots \end{pmatrix}}_{B}$$

$$= \underbrace{\begin{pmatrix} R_1 C_1 & R_1 C_2 & \cdots & R_1 C_p \\ R_2 C_1 & R_2 C_2 & \cdots & R_2 C_p \\ \vdots & & & \vdots \\ R_m C_1 & R_m C_2 & \cdots & R_m C_p \end{pmatrix}}_{AB}$$

This is a $m \times p$ matrix.

Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 3 & 3 \\ 0 & 2 & 0 \end{pmatrix}$$

Calculate AB

Sol^n

$$AB = \begin{pmatrix} 1 & 14 & 7 \\ 1 & 35 & 19 \end{pmatrix}$$

$$(1\ \ 2\ \ 3)\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -1+2+0 = 1$$

$$(1\ \ 2\ \ 3)\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = 1.2+2.3+3.2 = 14$$

$$(1\ \ 2\ \ 3)\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = 1.1+2.3+3.0 = 7$$