Review of important terminology

- $f(x)$ is continuous at $x=c$ if $f(c)$ is defined and

$$\lim_{x\to c} f(x) = f(c).$$

- $f(x)$ is differentiable at $x=c$ if $f(c)$ is defined and the limit

$$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$

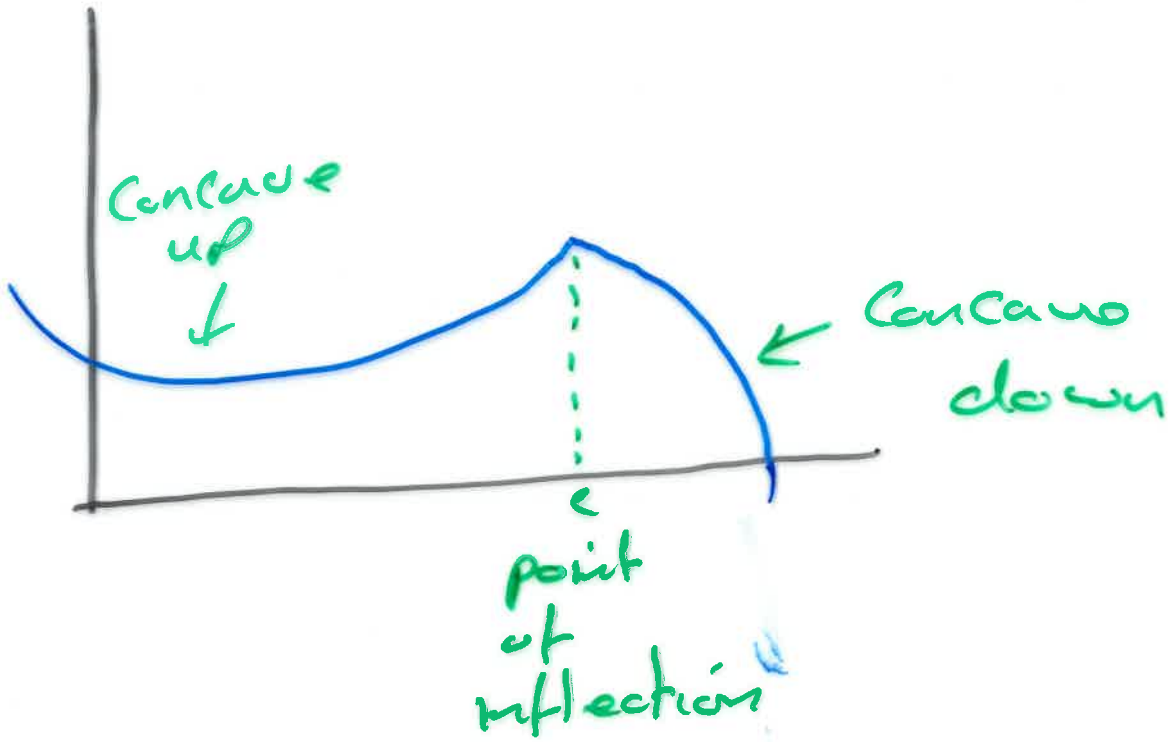
exists.

- $x=c$ is a critical point of $f(x)$ if either $f'(c)$ does not exist or $f'(c)=0$.

- $f(x)$ is increasing on (a,b) if $f'(x) \geq 0$ for all $x \in (a,b)$.

- $f(x)$ is concave up on (a,b) if $f''(x) \geq 0$ for $x \in (a,b)$.

- $x=c$ is a point of inflection of $f(x)$ if the concavity changes at $x=c$.



Theorem If $f(x)$ is differentiable at $x = c$ then $f(x)$ is continuous at $x = c$.

Proof

Suppose $f(x)$ is differentiable at $x = c$.

This means

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = l \quad \text{exists.}$$

This implies

$$\left(\lim_{h\to 0} h\right)\left(\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}\right) = \left(\lim_{h\to 0} h\right) l .$$

This implies

$$\lim_{h\to 0}\left(\cancel{h} \cdot \frac{f(c+h)-f(c)}{\cancel{h}}\right) = 0 .$$

This implies

$$\lim_{h\to 0} f(c+h) - \lim_{h\to 0} f(c) = 0 .$$

This implies

$$\lim_{h\to 0} f(c+h) = f(c) .$$

This implies

$$\lim_{x\to c} f(x) = f(c) .$$

QED

<u>Example</u> Find all possible values a, b such that

$$f(x) = \begin{cases} x^2+x+1 , & x \geq 1 \\ ax+b , & x < 1 \end{cases}$$

is differentiable at all points.

<u>Sol^n</u> $f(x)$ is "clearly" differentiable at all points $x \neq 1$.

Need to choose a, b such that $f(x)$ is differentiable at $x = 1$.

In particular, $f(x)$ must be continuous at $x=1$. i.e.

$$\lim_{x \to 1} f(x) = f(1).$$

This means

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

This means

$$\lim_{x \to 1^-} ax + b = 3$$

So $\boxed{a + b = 3}$

Now $f'(1)$ exists means

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \text{ exists.}$$

$\Longleftrightarrow$

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$$

$\Longleftrightarrow$

$q'(1) = p'(1)$

where $q(x) = ax + b$ where $p(x) = x^2 + x + 1$

$q'(1) = a \qquad p'(1) = 3$

$$a = 3$$

and $b = 0$

Rolle's Theorem

Suppose $f(x)$

- is continuous ~~on~~ on $[a, b]$
- differentiable on (a, b)
- and $f(a) = f(b)$.

Then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.