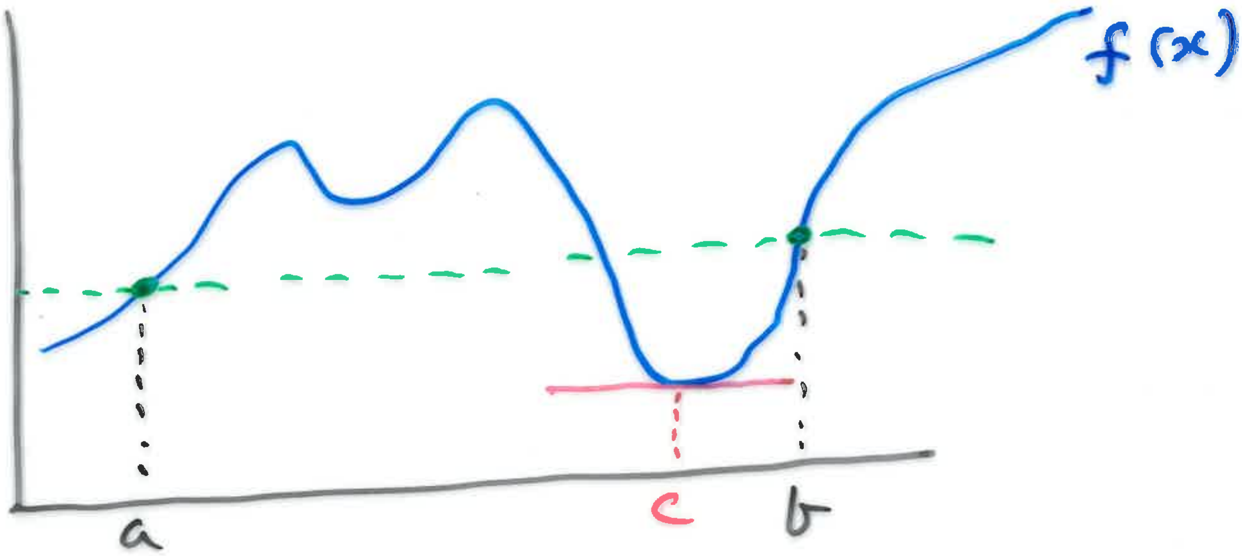


Theorem (Rolle)

If $f(x)$ is continuous at all points in $[a, b]$, and if $f(x)$ is differentiable at all points in (a, b), and if $f(a) = f(b)$ then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

<u>Exercise</u> Prove that there is exactly one solution to

$$x^3 + x + 1 = 0 \qquad (*)$$

<u>Soln</u>

Let $f(x) = x^3 + x + 1$.
Note that f is differentiable at all points in $\mathbb{R}$.

$f(-1) < 0$

$f(0) > 0$

So the Intermediate Value Theorem implies that there is at least one $c \in [-1, 0]$ such that $f(c) = 0$. So there is <u>at least</u> one solution to $(*)$.

$f'(x) = 3x^2 + 1.$

Note that $f'(x) > 0$ for all $x \in \mathbb{R}$.

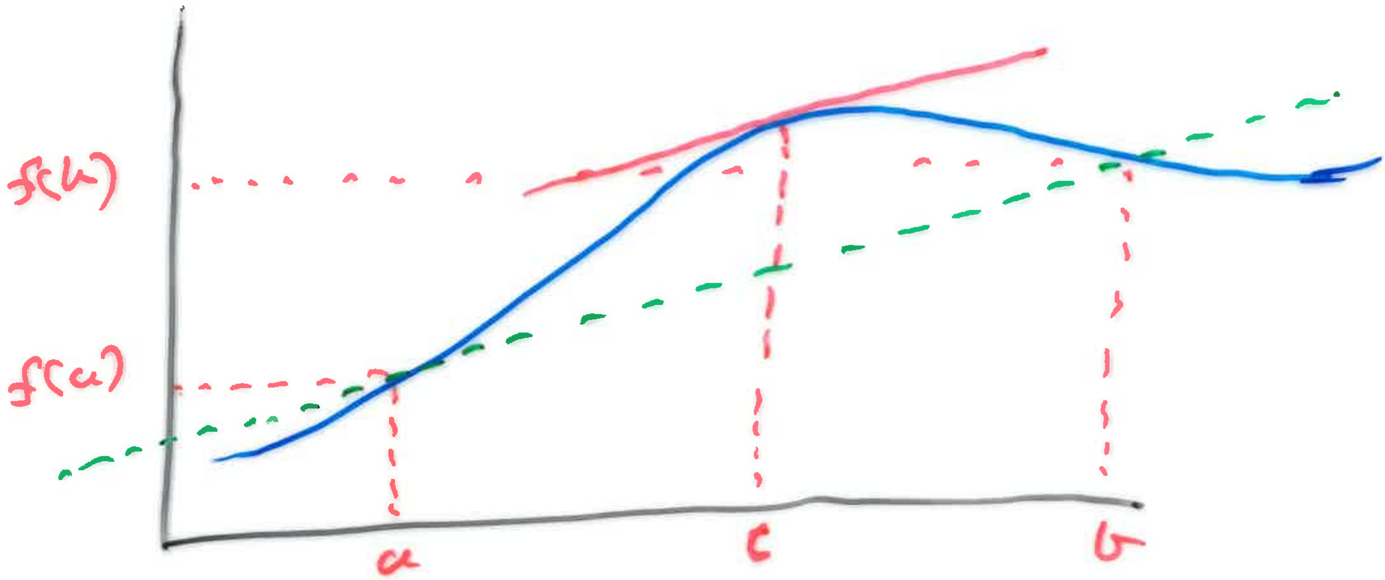
If there were two solutions to (*), say

$$f(a) = f(b) = 0,$$

then by Rolle's theorem we'd have $f'(x) = 0$ for some x. But $f'(x) > 0$ for all x.

Hence (*) has exactly one solution.

The Mean Value Theorem



<u>Theorem</u> Suppose that $f : [a,b] \longrightarrow \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then there exists at least one point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

Logarithms & Exponents

$4^2 = 16$	$\log_4 16 = 2$
$4^3 = 64$	$\log_4 64 = 3$
$4^{\frac{1}{2}} = \sqrt{4} = 2$	$\log_4 2 = \frac{1}{2}$
$4^{-2} = \frac{1}{4^2} = \frac{1}{16}$	$\log_4 \frac{1}{16} = -2$
$4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8$	$\log_4 8 = \frac{3}{2}$
$4^{\frac{5}{2}} = \left(4^{\frac{1}{2}}\right)^5 = 2^5 = 32$	$\log_4 32 = \frac{5}{2}$

We can make sense of 4^n when n is an integer $n = 0, \pm 1, \pm 2, \ldots$ and when n is a fraction

$$4^{\frac{p}{q}} = \left(\sqrt[q]{4}\right)^p$$

Terminology: fraction of the form $\frac{p}{q}$ with p, q integers are called rational numbers.

What does $3^{\sqrt{2}}$ mean?

We'd like to treat exponents (and logarithms) carefully so that we have a meaning to $3^{\sqrt{2}}$.

Is $(\sqrt{2})^{\sqrt{2}}$ rational or not?

I have no idea!

Properties of logarithms

$$\boxed{y = a^x \iff \log_a y = x}$$

$$\boxed{(a^m)^n = a^{mn}} \quad (1)$$

$$\boxed{a^m a^n = a^{m+n}} \quad (2)$$

Suppose

$$u = a^m, \quad v = a^n$$

$$\log_a(uv) = \log_a(a^m a^n)$$

$$\overset{(2)}{=} \log_a(a^{m+n})$$

$$= m + n$$

$$= \log_a(u) + \log_a(v)$$

"Thus"

$$\boxed{\log_a(uv) = \log_a(u) + \log_a(v)} \quad 2'$$

Now let $u = a^m$

Then

$$\log_a u^n = \log_a((a^m)^n)$$

$$\underset{1}{=} \log_a a^{mn}$$

$$= mn$$

$$= n \log_a u.$$

"Thus"

$$\boxed{\log_a(u^n) = n\log_a(u)} \quad 1'$$