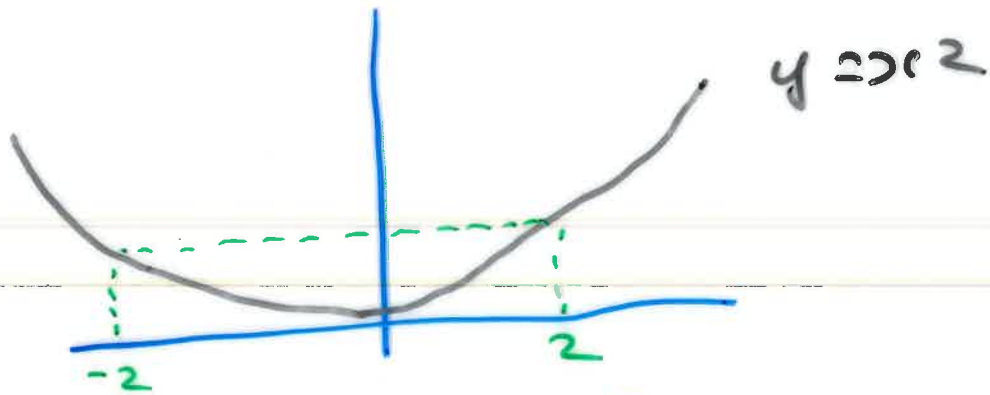
A function $f : D \to \mathbb{R}$ is said to be injective if $f(x_1) \neq f(x_2)$ for all $x_1, x_2 \in D$.

Example (a) $f(x) = x^2$; $D = \mathbb{R}$

This is not injective because, for instance, $f(2) = f(-2)$.



Example (b) $f(x) = x^3 - 2$.

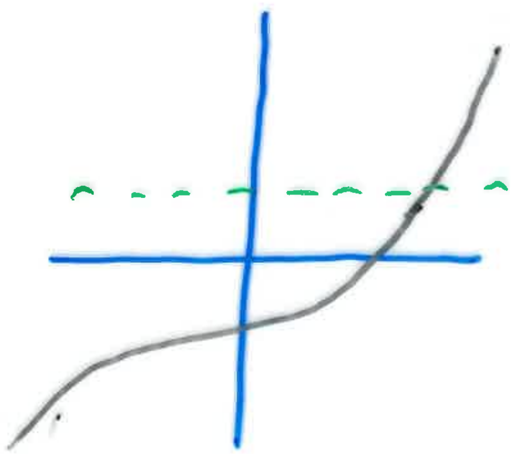
if $f(x_1) = f(x_2)$

then $x_1^3 - 2 = x_2^3 - 2$

and $x_1^3 = x_2^3$

and $x_1 = x_2$.

so $f(x)$ is injective



Suppose that $f : D \to \mathbb{R}$ is an injective function. Then the inverse function f^{-1} is defined by the rule

$f^{-1}(y) = x$ precisely when $y = f(x)$.

The domain of $f(x)$ is equal to the range of f.

Example Find the inverse f^{-1} of the function $f(x) = x^3 - 2$, with Domain$(f) = \mathbb{R}$.

Sol^n

$$y = x^3 - 2$$

$$y + 2 = x^3$$

$$\sqrt[3]{y+2} = x$$

So $f^{-1}(x) = \sqrt[3]{x+2}$ with domain $D = \mathbb{R}$.

Observe

$$\boxed{f^{-1}(f(x)) = x} \qquad (*)$$

Proposition (Derivative of an inverse function)

Suppose

$f'(f^{-1}(x)) \neq 0$ for any x.

Then

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Proof $\quad f^{-1}(f(x)) = x \qquad (*)$

Differentiate both sides of $(*)$ (using chain rule on left-hand side)

$$(f^{-1})'(f(x)) \cdot f'(x) = 1$$

Now write $y = f(x)$ or

$x = f^{-1}(y)$, to get

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

QED

<u>Example</u> If $f(x) = x^3 - 2$

then $f^{-1}(x) = \sqrt[3]{x+2}$.

From the proposition

$$(f^{-1})'(x) = \frac{1}{3(\sqrt[3]{x+2})^2}$$

Exercise: check this by differentiating $f^{-1}(x) = \sqrt[3]{x+2}$.

Back to logarithms

From the definition of last lecture

$$\ln(x) : (0,\infty) \longrightarrow \mathbb{R}$$

we see that $\ln(x)$ is injective.

So $y = \ln(x)$ has an inverse function, which we denote by $\exp(x)$ or e^x.

So we have a meaning for $e^{\sqrt{2}}$.

Exercise: Use the above proposition to find $\frac{d}{dx} e^x$.

Ans: $\frac{d}{dx} e^x = e^x$.

Recall

$$\arcsin(y) = x$$

Means

$$\sin(x) = y.$$

We also write $\sin^{-1}(x)$ for $\arcsin(x)$.

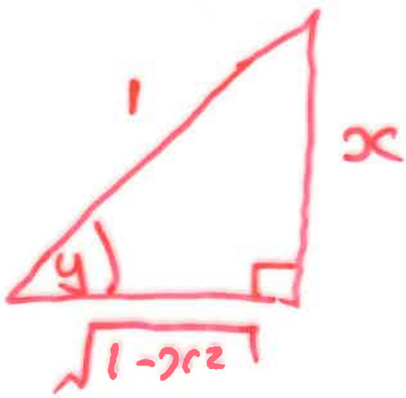
<u>Example</u> Find $\frac{dy}{dx}$ where $y = \sin^{-1}(x)$.

<u>Sol^n</u> $x = \sin(y)$

Differentiate both sides w.r.t. x:

$$1 = \cos(y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)}$$



$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$