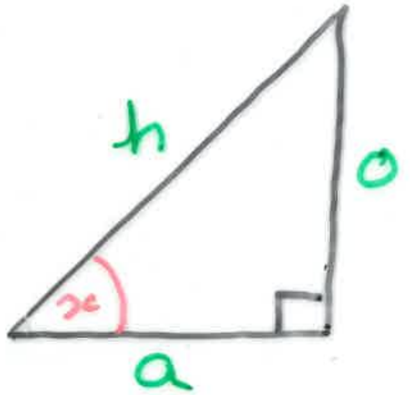
Limits of trigonometric functions

For $0 \leq x \leq \frac{\pi}{2}$ we have



$$\boxed{\sin(x) = \frac{o}{h} \qquad \cos(x) = \frac{a}{h}}$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{o}{a}$$

$$\sec(x) = \frac{1}{\cos(x)} \qquad \operatorname{cosec}(x) = \frac{1}{\sin(x)} \qquad \cot(x) = \frac{1}{\tan(x)}$$

useful identities:

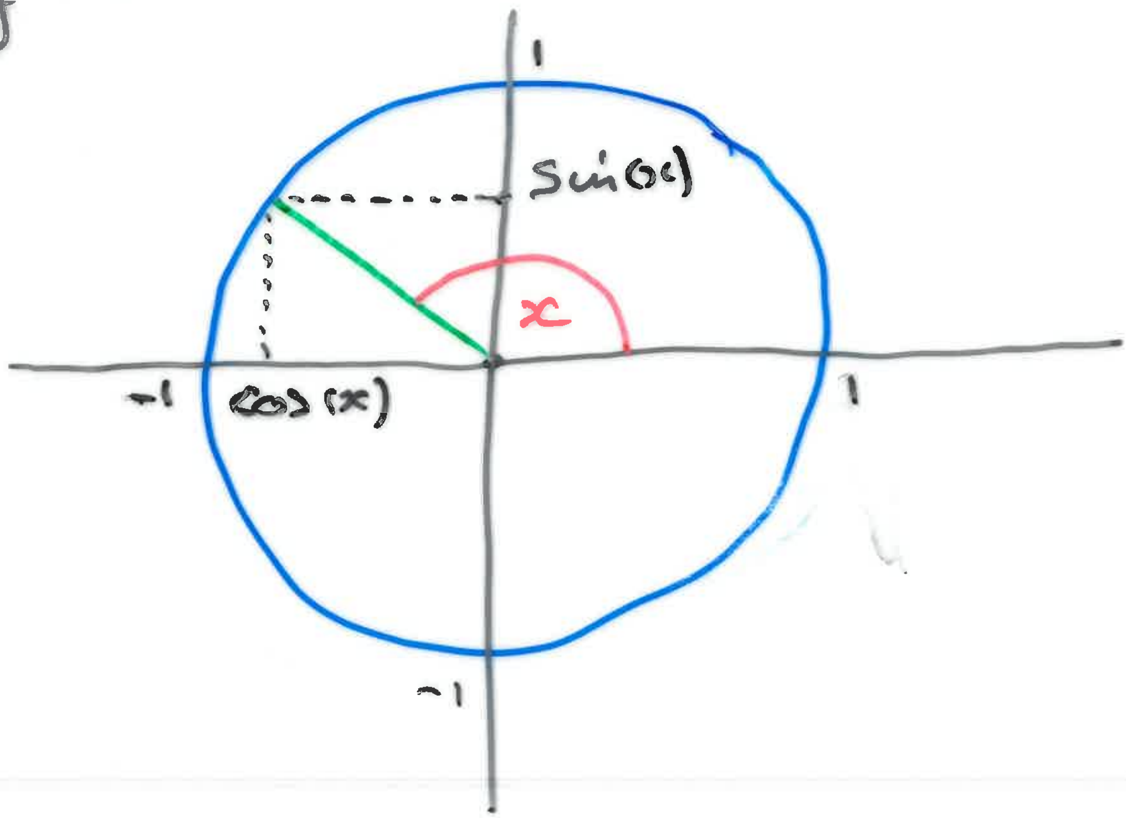
- $\cos^2(x) + \sin^2(x) = 1$

$$\frac{a^2}{h^2} + \frac{o^2}{h^2} = \frac{a^2+o^2}{h^2} = \frac{h^2}{h^2} = 1 .$$

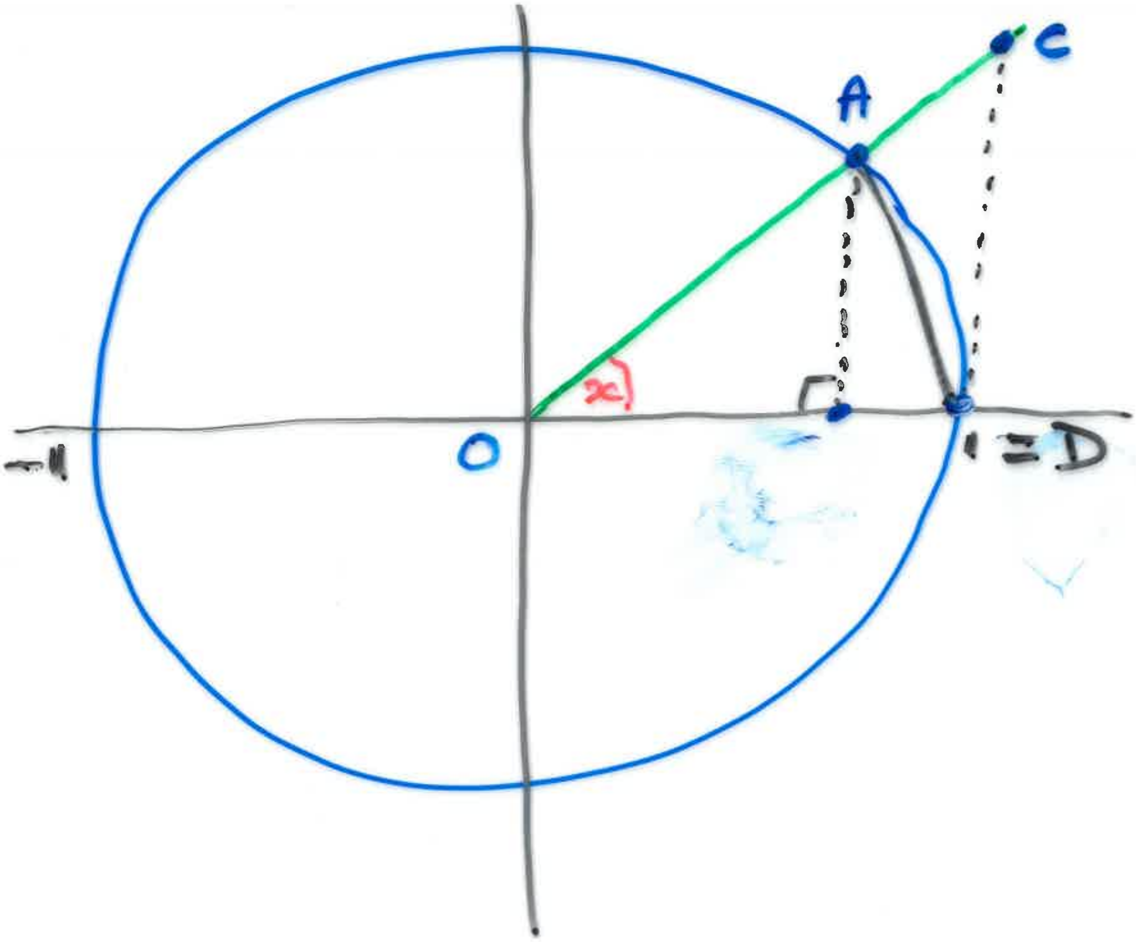
- $\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$

because ....

For arbitrary x we define $\sin(x)$, $\cos(x)$ using a circle of radius 1.



For small x :



Note:

area of triangle OAD $\leq$ area of sector OAD $\leq$ Area of triangle OCD

so

$$\frac{1}{2}\sin x \leq \frac{x}{2} \leq \frac{1}{2}\tan x$$

So

$$\frac{1}{2}\sin x \leq \frac{x}{2} \leq \frac{1}{2}\frac{\sin(x)}{\cos(x)}$$

$$\sin x \leq x \leq \frac{\sin x}{\cos x}$$

So

$$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$$

So for small $x \geq 0$

$$1 \geq \frac{\sin x}{x} \geq \cos(x)$$

Now, as $x \to 0$ we have

$\cos(x) \to 1.$

"Hence", by the Sandwich Lemma

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

strictly speaking, we also need to consider small $x \leq 0$ to complete the proof of this.

<u>Example</u> $\lim\limits_{x \to 0} \dfrac{1-\cos(x)}{x^2}$

<u>Sol^n</u>

$$\lim_{x \to 0} \frac{1-\cos(x)}{x^2}$$

$$= \lim_{x \to 0} \frac{(1-\cos(x))}{x^2} \frac{(1+\cos(x))}{(1+\cos(x))}$$

$$= \lim_{x \to 0} \frac{1-\cos^2 x}{x^2(1+\cos(x))}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x^2} \frac{1}{(1+\cos x)}$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 \frac{1}{(1+\cos x)}$$

$$= \frac{1}{2}.$$

Example Evaluate $\lim_{x \to 0} \dfrac{\tan 7x}{x}$.

Sol^n

$$\lim_{x \to 0} \frac{\tan 7x}{x}$$

$$= \lim_{x \to 0} \frac{\sin 7x}{x \cos 7x}$$

$$= \lim_{x \to 0} 7 \left(\frac{\sin 7x}{7x}\right) \frac{1}{\cos 7x}$$

$$= 7$$

L'Hôpital's Theorem

Suppose f and g are differentiable functions and that $g'(x) \neq 0$ for x in (a, b). Suppose that either

$$\lim_{x \to c} f(x) = 0 \quad \text{and} \quad \lim_{x \to c} g(x) = 0$$

or

$$\lim_{x \to c} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to c} g(x) = \pm\infty$$

for $c \in (a, b)$. Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

Example

Evaluate $\lim_{x \to 0} \frac{\tan x - x}{x^3}$.

Sol^n

$$\lim_{x \to 0} \frac{\tan x - x}{x^3}$$

$$\underset{\substack{\uparrow \\ \text{L'Hôpital}}}{=} \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2}$$

$$= \lim_{x \to 0} \frac{(\cos x)^{-2} - 1}{3x^2}$$

$$\underset{\substack{\uparrow \\ \text{L'Hôpital}}}{=} \lim_{x \to 0} \frac{+2(\cos x)^{-3} \sin x}{6x}$$

$$= \lim_{x \to 0} \frac{2}{6} \left(\frac{1}{\cos^3 x} \right) \frac{\sin x}{x}$$

$$= \frac{1}{3}.$$