# Topic 3

## Differential Equations

An equation involving a derivative, such as

$$\frac{dy}{dt} = ky \qquad (*)$$

where $k$ is a constant, and $y = f(t)$ is a function of $t$, is called a differential equation.

Are there any solutions to $(*)$?

Consider

$$y = e^{kt}$$

$$\frac{dy}{dt} = \frac{d}{dt} e^{kt} = k e^{kt} = ky$$

So $y = e^{kt}$ is one solution to $(*)$.

Another Solution is

$$y = 4e^{kt}$$

$$\frac{dy}{dt} = 4ke^{kt} = k4e^{kt} = ky$$

In fact, the function

$$y = Ae^{kt}$$

is a solution to $(*)$ for any constant $A$.

Question: Are there any other solutions to $(*)$?

Suppose $y = y(t)$ and $z = z(t)$ are both solutions to the differential equation $(*)$.

$$\frac{d}{dt}\left(\frac{y}{z}\right) = \frac{z'y - y'z}{z^2}$$

$$= \frac{kzy - kyz}{z^2}$$

$$= 0$$

So $\frac{y}{z}$ must be a constant,

Say $\frac{y}{z} = A$,

or $y = Az$.

Conclusion:

The only solutions to the diff. eq$^n$

$$\frac{dy}{dt} = ky \qquad (*)$$

are the functions

$$y = Ae^{kt}$$

with $A$ any constant.

Problem A cup of coffee in a room at 20°C cools from 80°C to 50°C in five minutes. How long will it take to cool to 40°C?

Sol^n

Newton: A hot object cools at a rate proportional to the excess of its temperature above room temperature.

$t$ = time, in minutes

$y(t)$ = temperature of coffee at time $t$.

$y(5) = 50$

$y(0) = 80$

Required to find the value of $t$ such that $y(t) = 40$.

Newton:

$$\frac{dy}{dt} = k(y-20)$$

Consider $z = y - 20$

$$z(0) = 60$$

$$z(5) = 30$$

$$\frac{dz}{dt} = \frac{d}{dt}(y-20) = \frac{d}{dt}y = k(y-20) = kz$$

So

$$\frac{dz}{dt} = kz \qquad (*)$$

Since $z$ satisfies $(*)$ we know

$$z = Ae^{kt}$$

where $A$, $k$ are constants.

Required to find $t$ such that $z(t) = 20$ (or $y(t) = 40$).

$z(0) = 60$

$60 = Ae^{k.0} = A$ $\qquad \boxed{A = 60}$

So $\boxed{z = 60e^{kt}}$

$z(5) = 30$

$$30 = 60\, e^{5k}$$

$$\boxed{\frac{1}{2} = e^{5k}}$$

Want $t$ such that

$$20 = z(t) = 60\, e^{kt}$$

$$\frac{1}{3} = e^{kt}$$

$$\frac{1}{3} = \left(e^{5k}\right)^{\frac{t}{5}}$$

$$\frac{1}{3} = \left(\frac{1}{2}\right)^{\frac{t}{5}}$$

$$\ln\left(\frac{1}{3}\right) = \ln\left(\left(\frac{1}{2}\right)^{\frac{t}{5}}\right)$$

$$\ln\left(\frac{1}{3}\right) = \frac{t}{5} \ln\left(\frac{1}{2}\right)$$

$$t = \frac{5 \ln\left(\frac{1}{3}\right)}{\ln\left(\frac{1}{2}\right)} \simeq 7.92 \text{ mins}$$