Anti-derivatives

A function $F(x)$ is called an anti-derivative of $f(x)$ if

$$\frac{d}{dx} F(x) = f(x).$$

Example An anti-derivative of $f(x) = \frac{1}{x+1}$ is

$$F(x) = \ln(x+1)$$

Example An anti-derivative of $f(x) = \frac{1}{3} x e^{x^2}$ is

$$F(x) = \frac{1}{6} e^{x^2}$$

check
$$\frac{d}{dx}\left(\frac{1}{6} e^{x^2} \right) = \frac{1}{6} e^{x^2} 2x = \frac{1}{3} x e^{x^2}$$

Lemma Any two anti-derivatives of $f(x)$ differ by a constant.

Proof If $F(x)$ and $G(x)$ are both anti-derivatives of $f(x)$ then

$$\frac{d}{dx}\left(F(x) - G(x)\right) = \frac{d}{dx}F(x) - \frac{d}{dx}G(x)$$

$$= f(x) - f(x)$$

$$= 0.$$

Hence $F(x) - G(x) = c$ a constant. $\boxtimes$

Notation If $F(x)$ is an anti-derivative of $f(x)$ we often write

$$\int f(x)\,dx = F(x) + c$$

Example

$$\int \cos(x)\,dx = \sin(x) + c$$

Example observations suggest that the rate of growth of a zebra mussel population is exponential in time.

Suppose

$$\frac{dy}{dt} = e^{2t}$$

where $y(t)$ is the number of mussels at time t (in days). If there are 150 mussels at time $t=0$, how many mussels will there be after 10 days?

Sol^n If $\dfrac{dy}{dt} = e^{2t}$ then

$$\boxed{y(t) = \frac{1}{2}e^{2t} + c}$$

$y(0) = 150$

Need to find $y(10)$.

$y(0) = 150 = \frac{1}{2}e^{0} + c$

$$\boxed{c = 149.5}$$

So $y(t) = \frac{1}{2}e^{2t} + 149.5$

At $t = 10$ days the population of mussels is

$y(10) = \frac{1}{2}e^{20} + 149.5 \simeq 2.4 \times 10^{8}$ mussels

Example The world's population was 2560 millions in 1950, and 3040 millions in 1960.

Thomas Malthus:

Assume that the growth rate of the population is proportional to the size of the world's population.

Questions:

a) Estimate the population in 1990.

b) " " " " 2060

Sol^n

Let $y(t)$ = population of the world at time t (years), in millions of people.

$$\boxed{\frac{dy}{dt} = k\, y}$$

where k is some constant.

from yesterday

$$\boxed{y = A e^{kt}}$$

$t = 0$ in year 1950

$y(0) = 2560 = A e^{0} = A$

$$\boxed{A = 2560}$$

$$\boxed{y = 2560\, e^{kt}}$$

$t = 10$ in 1960

$y(10) = 3040 = 2560\, e^{10k}$

$$\frac{3040}{2560} = e^{10k}$$

$$k = \frac{1}{10} \ln\left(\frac{3040}{2560}\right) \simeq 0.017185$$

So

$$\boxed{y(t) = 2560\, e^{0.017185\, t}}$$

In 1990, $t = 40$ and the population should have been

$$y(40) = 2560\, e^{0.017185 \times 40}$$

$$\simeq 5090 \text{ millions}$$

Actual population in 1990 was 5278 millions

In 2040, $t = 90$ and the world's population should be

$$y(90) = 2560\, e^{0.017185 \times 90}$$

$$\simeq 12021 \text{ millions.}$$