Seperable Differential Equations

A diff. equation is separable if it is of the form

$$f(y) \frac{dy}{dt} = g(t)$$

for functions $f(y)$, $g(t)$.

Example

$$y^2 \frac{dy}{dt} = t^2$$

is separable.

Malthusian Law

$$\frac{dy}{dt} = ky$$

is separable since we can write it as

$$\frac{1}{y} \frac{dy}{dt} = k.$$

model work's well for small populations y.

For large populations the following model seems to be better:

$$\boxed{\frac{dy}{dt} = ky - ly^2}$$

Logistic Model

where k, l are constants, and l is tiny compared to k.

The Logistic model is a separable diff. eqn since we can write it as

$$\frac{1}{ky - ly^2} \frac{dy}{dt} = 1$$

A separable eqn

$$\boxed{f(y)\,\frac{dy}{dt} = g(t)} \qquad (*)$$

can be rewritten as

$$\boxed{\frac{d}{dt}F(y) = g(t)}$$

where $F(y)$ is any anti-derivative of $f(y)$.

Consequently

$$\boxed{\boxed{F(y) = \int g(t)\,dt + C}}$$

solution to the diff. eqn $(*)$

Example Solve

$$y^2 \frac{dy}{dt} = t^2 , \quad y(0) = 7$$

Sol^n

$$\int y^2\, dy = \int t^2\, dt + c$$

$$\frac{1}{3} y^3 = \frac{1}{3} t^3 + c$$

or

$$y^3 = t^3 + c$$

$$\boxed{y = (t^3 + c)^{\frac{1}{3}}}$$ Sol^n

$y(0) = 7$

$$(0^3 + c)^{\frac{1}{3}} = 7$$

$$c = 7^3$$

$$\boxed{y = (t^3 + 7^3)^{\frac{1}{3}}}$$ Sol^n

Example Solve

$$e^{y}\frac{dy}{dt} - t - t^{3} = 0, \quad y(0) = 1.$$

Sol^n

$$e^{y}\frac{dy}{dt} = t^{3} + t$$

$$\int e^{y}\,dy = \int (t^{3}+t)\,dt + c$$

$$e^{y} = \frac{t^{4}}{4} + \frac{t^{2}}{2} + c$$

$$\ln(e^{y}) = \ln\left(\frac{t^{4}}{4} + \frac{t^{2}}{2} + c\right)$$

$$\boxed{y = \ln\left(\frac{t^{4}}{4} + \frac{t^{2}}{2} + c\right)}$$

$y(0) = 1$

$$1 = \ln(0 + 0 + c)$$

$$c = e$$

$$\boxed{y = \ln\left(\frac{t^{4}}{4} + \frac{t^{2}}{2} + e\right)}$$ Solution

Example Solve

$$\frac{dy}{dt} = ky - ly^2 .$$

Sol^n

$$\frac{1}{ky - ly^2} \frac{dy}{dt} = 1$$

$$\int \frac{1}{ky - ly^2}\, dy = \int dt + c$$

RHS : $\int dt = t + c$

LHS

suppose

$$f(y) = \frac{1}{ky - ly^2} = \frac{1}{y(k-ly)}$$

$$f(y) = \frac{1}{y(k-ly)} = \frac{A}{y} + \frac{B}{k-ly}$$

Then we'd have

$$\int f(y)\,dy = \int \frac{A}{y} + \frac{B}{k - ly}\,dy$$

$$= A\ln(y) - \frac{B}{l}\ln(k - ly).$$