World Population

Let $y(t)$ be the population at time t.

The Logistic model is :

$$\boxed{\frac{dy}{dt} = ky - ly^2} \qquad (*)$$

where k, l are constants, and l is much smaller than k.

Equation $(*)$ is separable :

$$\boxed{\frac{1}{ky - ly^2}\frac{dy}{dt} = 1}$$

The solution is given by

$$\boxed{\int \frac{1}{ky - ly^2}\, dy = \int 1\, dt} \qquad 1$$

Suppose

$$\frac{1}{y(k-ly)} = \frac{A}{y} + \frac{B}{k-ly}$$

Let's find A & B.

$$\frac{1}{y(k-ly)} = \frac{A(k-ly)+By}{y(k-ly)}$$

So

$$1 = A(k-ly) + By$$

$$\boxed{1 = Ak + (B - Al)y}$$

Need

$$\boxed{\begin{aligned} 1 &= Ak \\ 0 &= B - Al \end{aligned}}$$

So

$$\boxed{\begin{aligned} A &= \frac{1}{k} \\ B &= \frac{l}{k} \end{aligned}}$$

Thus

$$\frac{1}{ky-ly^2} = \frac{1}{ky} + \frac{l}{k^2-kly}$$

$$\int \frac{1}{ky-ly^2}\,dy = \frac{1}{k}\int \frac{k}{ky}\,dx + \int \frac{l}{k^2-kly}\,dy$$

$$= \frac{1}{k}\int \frac{k}{ky}\,dy - \frac{1}{k}\int \frac{-kl}{k^2-kly}\,dy$$

$$= \frac{1}{k}\ln(ky) - \frac{1}{k}\ln(k^2-kly)$$

Thus an anti-derivative of $f(y) = \frac{1}{ky - ly^2}$ is

$$F(y) = \frac{1}{k} \ln(ky) - \frac{1}{k} \ln(k^2 - kly)$$

$$F(y) = \frac{1}{k}\left(\ln(ky) - \ln(k^2 - kly)\right)$$

$$F(y) = \frac{1}{k} \ln\left(\frac{ky}{k^2 - kly}\right)$$

$$F(y) = \frac{1}{k} \ln\left(\frac{y}{k - ly}\right)$$

Equation (1) becomes

$$\frac{1}{k} \ln\left(\frac{y}{k - ly}\right) = t$$

$$\Leftrightarrow \quad kt = \ln\left(\frac{y}{k - ly}\right)$$

$$\Leftrightarrow \quad e^{kt} = \frac{y}{k - ly}$$

$$\Leftrightarrow \quad (k - ly)e^{kt} = y$$

$$\Leftrightarrow \quad k e^{kt} = y(1 + l e^{kt})$$

$$\boxed{y = \frac{k e^{kt}}{1 + l e^{kt}}}$$

So $y \to \frac{k}{l}$ as $t \to \infty$

Concluding: The Logistic model implies that the population of the world will tend to some constant population $\frac{k}{l}$.

We can estimate k and l from past populations.

An estimate, using 1950, 1960 and 1970 the limiting population is

$y(t) \to \frac{k}{l} \simeq 9.86$ billion.