Consider $g(x) = \dfrac{x^6 - 1}{x - 1}$.

$\text{Domain}(g) = \mathbb{R} \setminus \{1\}$

$\text{Codomain}(g) = \mathbb{R}$.

$$g(0.9) = \frac{(0.9)^6 - 1}{0.9 - 1} = 4.68559$$

$$g(1.1) = \frac{(1.1)^6 - 1}{1.1 - 1} = 7.71561$$

$$g(0.99) = \frac{(0.99)^6 - 1}{0.99 - 1} = 5.85198506$$

$$g(0.999) = \frac{(0.999)^6 - 1}{0.999 - 1} = 5.985019985$$

We write

$$\lim_{x \to 1} g(x) = 6$$

to mean that for all real numbers x "close to 1", but distinct to 1, the value of $g(x)$ gets "close to 6".

Example Evaluate

$$\lim_{x \to 0} \frac{\sqrt{4+x} - 2}{x}$$

Sol^n

for $x \neq 0$ and close to 0

$$\frac{\sqrt{4+x} - 2}{x} \frac{(\sqrt{4+x} + 2)}{(\sqrt{4+x} + 2)}$$

$$= \frac{4+x-4}{x(\sqrt{4+x}+2)}$$

$$= \frac{\cancel{x}}{\cancel{x}(\sqrt{4+x}+2)} = \frac{1}{\sqrt{4+x}+2}$$

$$\lim_{x \to 0} \frac{\sqrt{4+x} - 2}{x} = \lim_{x \to 0} \frac{1}{\sqrt{4+x}+2} = \frac{1}{4}.$$

Example Evaluate

$$\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{4}{x^2-4}\right)$$

Sol^n

For $x \neq 2$ and x close to 2

$$\frac{1}{x-2}-\frac{4}{x^2-4} = \frac{1}{x-2}-\frac{4}{(x+2)(x-2)}$$

$$= \frac{(x+2)-4}{(x+2)(x-2)}$$

$$= \frac{\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$$

$$= \frac{1}{x+2}.$$

$$\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{4}{x^2-4}\right) = \lim_{x\to 2}\left(\frac{1}{x+2}\right) = \frac{1}{4}.$$

Recall $|-3| = 3$

$|3| = 3$

Example

$$\lim_{x \to 0} \frac{x}{|x-1| - |x+1|}$$

Sol^n For $x \neq 0$ and x sufficiently close to 0

$$\frac{x}{|x-1| - |x+1|}$$

$$= \frac{x}{(1-x) - (x+1)}$$

$$= \frac{\cancel{x}}{-2\cancel{x}} = -\frac{1}{2}.$$

$$\lim_{x \to 0} \frac{x}{|x-1| - |x+1|} = -\frac{1}{2}.$$

Correct definition of a limit.

We write

$$\lim_{x \to 1} g(x) = 6$$

to mean that

for any real number $\varepsilon > 0$

there exist a number $\delta > 0$

such that

$0 < |x-1| < \delta$

implies

$$|g(x) - 6| < \varepsilon$$