Proposition

$$\lim_{x \to 1} \frac{x^6 - 1}{x - 1} = 6.$$

Proof Let $\varepsilon > 0$. We can choose δ such that $|x-1| < \delta$ implies $|x-1| < \frac{\varepsilon}{5}$, $|x^2-1| < \frac{\varepsilon}{5}$, $\dots$, $|x^5-1| < \frac{\varepsilon}{5}$.

Then

$$0 < |x-1| < \delta$$

implies

$$\left| \frac{x^6-1}{x-1} - 6 \right| = \left| \frac{(x-1)(1+x+x^2+x^3+x^4+x^5)}{(x-1)} - 6 \right|$$

$$= |1 + x + \cdots + x^5 - 6|$$

$$= |(1-1) + (x-1) + (x^2-1) + (x^3-1) + (x^4-1) + (x^5-1)|$$

$$\leq |1-1| + |x-1| + |x^2-1| + \cdots + |x^5-1|$$

$$< 0 + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5}$$

$$= \varepsilon.$$

QED

Proposition Suppose

$\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$

both exist. Then

i) $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

ii) For any $k \in \mathbb{R}$

$$\lim_{x \to a} k f(x) = k \lim_{x \to a} f(x).$$

iii) $\lim_{x \to a} (f(x) g(x)) =$

$$\left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$$

iv) $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

provided $\lim_{x \to a} g(x) \neq 0$.

Example

$$\lim_{x \to 2} \frac{x^2 + 4x + 5}{x^2 + 5}$$

$$\underset{(iv)}{=} \frac{\lim_{x \to 2} (x^2 + 4x + 5)}{\lim_{x \to 2} (x^2 + 5)}$$

$$\underset{(i)}{=} \frac{\lim_{x \to 2} x^2 + \lim_{x \to 2} 4x + \lim_{x \to 2} 5}{\lim_{x \to 2} x^2 + \lim_{x \to 2} 5}$$

$$\underset{(ii) \& (iii)}{=} \frac{\left(\lim_{x \to 2} x\right)^2 + 4 \lim_{x \to 2} x + \lim_{x \to 2} 5}{\left(\lim_{x \to 2} x\right)^2 + \lim_{x \to 2} 5}$$

$$= \frac{2^2 + 4 \cdot 2 + 5}{2^2 + 5}$$

$$= \frac{17}{9}.$$