Theorem
Proposition
Lemma

true statement with proof

Conjecture

statement for which there is currently no proof.

Sandwich Lemma

Suppose

$$f(x) \le g(x) \le h(x)$$

for all x sufficiently near a (except for possibly $x = a$). Suppose also

$$\lim_{x \to a} f(x) = \ell = \lim_{x \to a} h(x)$$

Then

$$\lim_{x \to a} g(x) = \ell .$$

Example Evaluate

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$

Sol^n

$g(x) = x^2 \sin\left(\frac{1}{x}\right)$

$f(x) = -x^2$

$h(x) = x^2$

For x near 0 and $x \neq 0$ we have

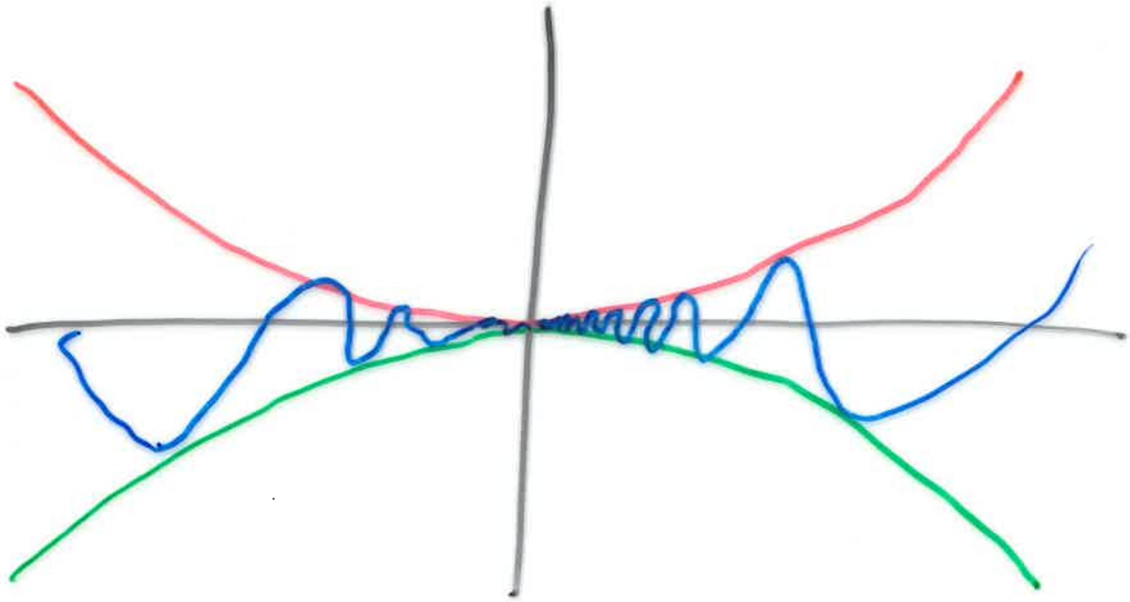
$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2 .$$

$$\lim_{x \to 0} -x^2 = 0$$

$$\lim_{x \to 0} x^2 = 0$$

Thus, by the Sandwich Lemma,

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$



Note $g(x) = x^2 \sin\left(\frac{1}{x}\right)$ is not defined at $x=0$.

$\text{Domain}(g(x)) = \mathbb{R} \setminus \{0\}$.

Left-hand and right-hand limits

We write

$$\lim_{x \to a-} f(x) = l$$

to mean that $f(x)$ is close to l for all x sufficiently close to a and strictly less than a.

Example Let $f(t)$ denote the car park charge in Dublin Airport for t hours of parking.

More precisely

$$f(t) = \begin{cases} 0 & 0 \leq t < 0.25 \\ 5 & 0.25 \leq t \leq 1 \\ 5t & 1 < t < \infty \end{cases}$$

$$\lim_{t \to 0.25^-} f(t) = 0$$

$$\lim_{t \to 0.25^+} f(t) = 5$$

$$f(0.25) = 5$$

Example Let

$$f(x) = \begin{cases} x^2+1 & \text{if } x < 3 \\ x+8 & \text{if } x \geq 3 \end{cases}$$

$\text{Domain}(f) = \mathbb{R}$

$\text{codomain}(f) = \mathbb{R}$

$f : \mathbb{R} \to \mathbb{R}$

$$\lim_{x \to 3^-} f(x) = 10$$

$$\lim_{x \to 3^+} f(x) = 11$$

$f(3) = 11$

Proposition

$$\lim_{x \to a} f(x) = \ell$$

if and only if

$$\lim_{x \to a^-} f(x) = \ell = \lim_{x \to a^+} f(x).$$

If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

then $\lim_{x \to a} f(x)$ does not

exist.

Example Let

$$f(x) = \begin{cases} x^2+1 & \text{if } x < 3 \\ x+c & \text{if } x \geq 3 \end{cases}$$

where c is some constant.

For what value of c does

$$\lim_{x \to 3} f(x)$$

exist?

Sol^n

$$\lim_{x \to 3-} f(x) = 10$$

$$\lim_{x \to 3+} f(x) = 3+c$$

By the above proposition we need $10 = 3+c$ for

$\lim_{x \to 3} f(x)$ to exist.

So need $c = 7$.