Intermediate Value Theorem

Suppose

$$y = f(x)$$

is a real valued function which is continuous at all points x in the interval

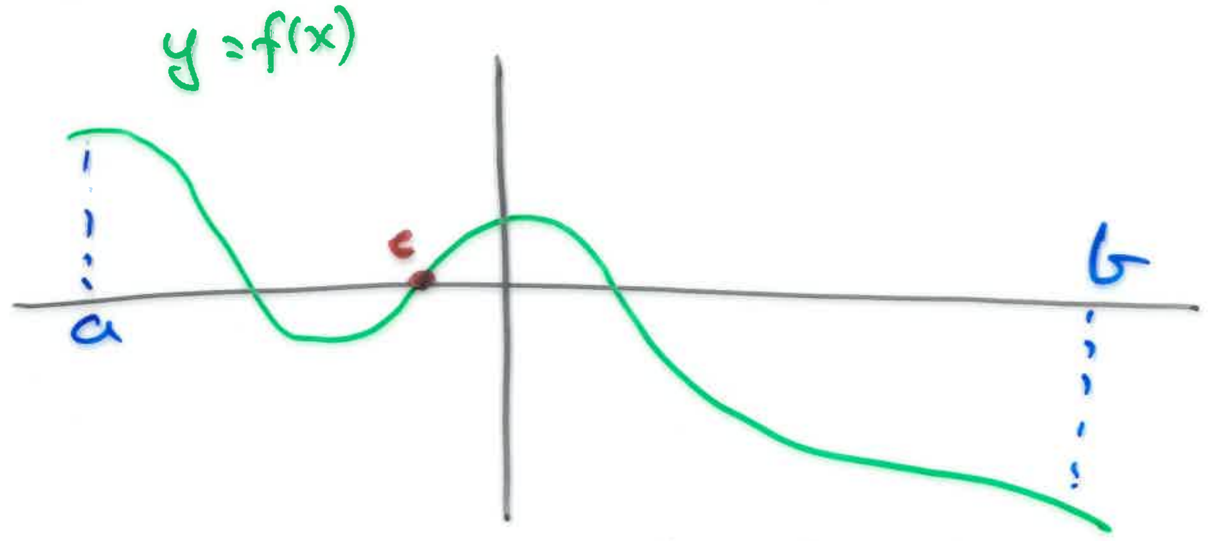
$$a \le x \le b.$$

Suppose also that

$$f(a)\,f(b) \le 0.$$

Then there exists at least one value c in the interval $a \le c \le b$ such that

$$f(c) = 0.$$



Example Prove that

$$x^3 - x - 1 = 0 \qquad (*)$$

has a solution in the range $1 \le x \le 2$.

Proof

Let $f(x) = x^3 - x - 1$.

"clearly" this function $f(x)$ is continuous.

Let $a = 1$, $b = 2$.

$$\left.\begin{array}{l} f(1) < 0 \\ f(2) > 0 \end{array}\right\} \text{ so } f(1)f(2) < 0$$

The IVT says that there exists at least one value c, $1 \le c \le 2$, such that

$$f(c) = 0.$$

i.e. c is a solution to $(*)$.

QED

<u>Example</u> Show that

$$x^3 - 4x + 1 = 0$$

has three real solutions, and find approximations to them.

<u>Sol^n</u>

Let $f(x) = x^3 - 4x + 1$

$f(0) > 0$

$f(1) < 0$

$f(2) > 0$

$f(-1) > 0$

$f(3)$ forget it!

$f(-3) < 0$

IVT says that $f(x) = 0$ for some

$x \in [0, 1]$

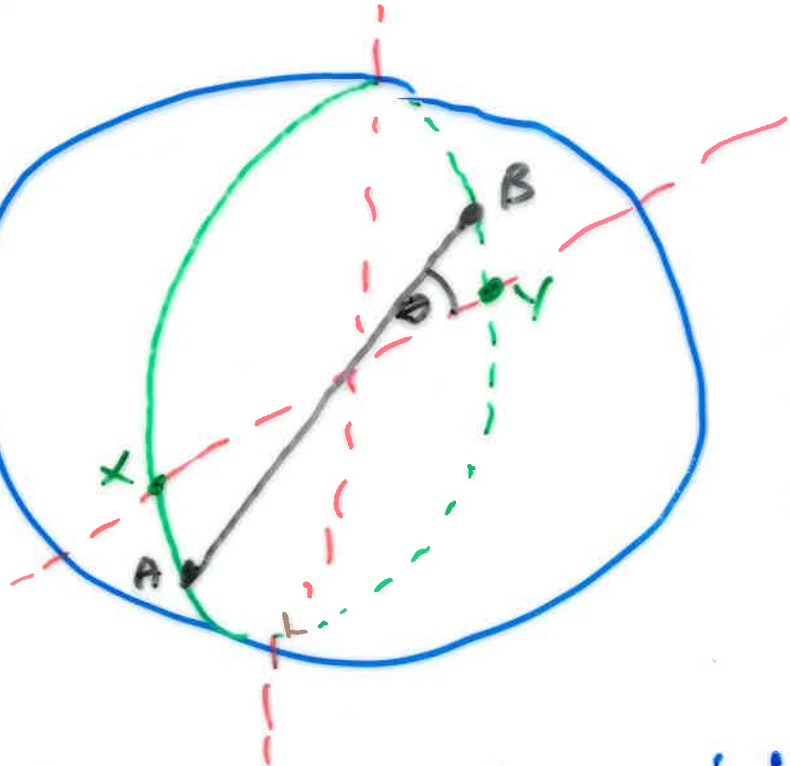
$x \in [1, 2]$

$x \in [-3, -1]$.

So there are at least 3 solutions.

(FACT: A polynomial of degree 3 has at most 3 solutions.)

Application of IVT



Take any great circle on the Earth.

FACT There exist opposite points on your great circle with equal air pressure.

Explanation of this fact

Consider

$f(\theta)$ = air pressure at A
— air pressure at B.

Note: $f(\theta)$ is a continuous function of θ.

I want to prove that for some value of θ, pressure at A = pressure at B.

i.e. I want to prove that

$$f(\theta) = 0$$

for some $\theta \in [0, \pi]$

If $f(0) = 0$, or if $f(\pi) = 0$ then we'd have $f(0)f(\pi) \leq 0$.

Suppose

$$f(0) \neq 0 \text{ and } f(\pi) \neq 0.$$

Note: $f(0)f(\pi) < 0$

$f(0)$ = air pressure at x
  − air pressure at y

$f(\pi)$ = air pressure at y
  − air pressure at x.

So IVT says there is some $\theta \in [0, \pi]$ such that

$$f(\theta) = 0.$$

QED