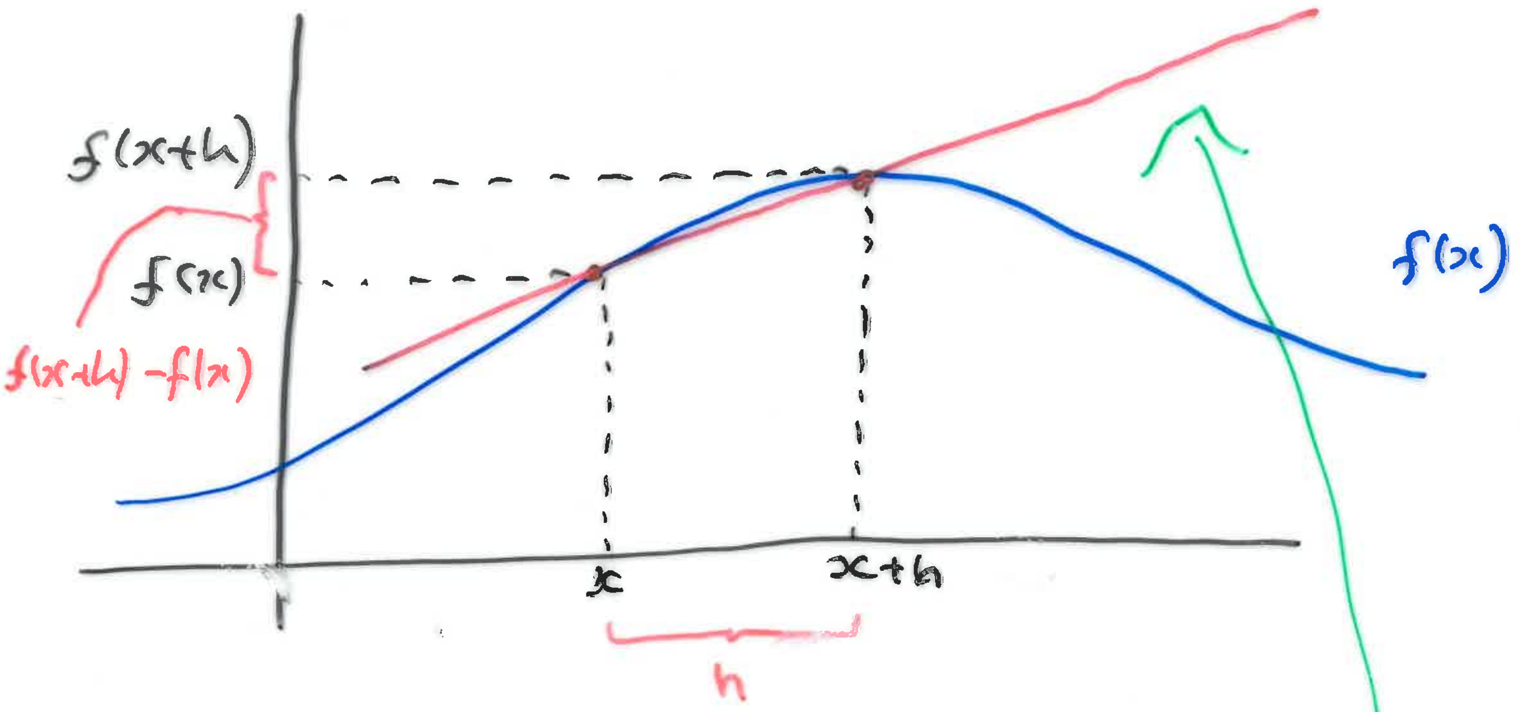
Derivatives



$$\frac{f(x+h) - f(x)}{h} = \text{slope of red line}$$

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}} = \text{slope of the tangent to curve } y = f(x) \text{ at the point } x.$$

Definition

Remember:

$$\frac{d}{dx} f(x) = f'(x)$$

Rules of differentiation

$$\boxed{\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)}$$

Sum Rule

Proof

LHS =

$$\lim_{h\to 0} \frac{\left(f(x+h)+g(x+h)\right) - \left(f(x)+g(x)\right)}{h}$$

$$= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right)$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

Example

$$\frac{d}{dx}\left(x^{\frac{3}{2}} + \sin(x)\right)$$

$$= \frac{d}{dx}x^{\frac{3}{2}} + \frac{d}{dx}\sin(x)$$

$$= \frac{3}{2}x^{\frac{1}{2}} + \cos(x).$$

$$\boxed{\frac{d}{dx}\left(kf(x)\right) = k\frac{d}{dx}f(x)}$$

Scalar product rule
(here k is any constant)

Example

$$\frac{d}{dx}\left(3e^{x}\right) = 3\frac{d}{dx}\left(e^{x}\right)$$

$$= 3e^{x}.$$

$$\boxed{\frac{d}{dx}\left(f(x)\,g(x)\right) = \left[\frac{d}{dx}f(x)\right]g(x) + f(x)\left[\frac{d}{dx}g(x)\right]}$$

Product Rule

Example

$$\frac{d}{dx}\left(x^2 \sin x\right)$$

$$= \left[\frac{d}{dx}x^2\right]\sin x + x^2\left[\frac{d}{dx}\sin x\right]$$

$$= 2x \sin x + x^2 \cos x$$

Example $y = (x^2+1)(x^3+2)$

$$\frac{dy}{dx} = \left(\frac{d}{dx}(x^2+1)\right)(x^3+2) + (x^2+1)\frac{d}{dx}(x^3+2)$$

$$= 2x(x^3+2) + (x^2+1)3x^2$$

$$= 5x^4 + 3x^2 + 4x$$

Chain Rule

Given functions $f(x)$ and $g(x)$ we can consider the composite function

$$y = g(f(x)).$$

$$\boxed{\frac{dy}{dx} = g'(f(x))\, f'(x)}$$

Chain Rule

Example $y = \sin(x^2)$

$f(x) = x^2 \qquad g(x) = \sin(x)$

Sol^n

$$\frac{dy}{dx} = \cos(x^2)\, 2x$$

$$= 2x \cos(x^2).$$

Example

$$y = (x^2 - x + 1)^7$$

$$\frac{dy}{dx} = 7(x^2 - x + 1)^6 (2x - 1)$$

Example

$$y = \sqrt{x^2 + 1}$$

$$y = (x^2 + 1)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} 2x$$

$$= \frac{x}{\sqrt{x^2 + 1}}$$

Quotient Rule

$$y = \frac{f(x)}{g(x)}$$

$$y = f(x)\,(g(x))^{-1}$$

$$\frac{dy}{dx} = f'(x)\,(g(x))^{-1} + f(x)\,\frac{d}{dx}\left((g(x))^{-1}\right)$$

$$\frac{dy}{dx} = f'(x)\,(g(x))^{-1} - f(x)\,(g(x))^{-2}\,g'(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{g(x)} - \frac{f(x)\,g'(x)}{(g(x))^2}$$

$$\boxed{\frac{dy}{dx} = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g(x)^2}}$$

QUOTIEN Rule