MA2286: Tutorial Problems 2020-21

For those questions taken from the Schaum Outline Series book *Advanced Calculus* by M. Spiegel the question number in the book is given. The book provides worked solutions for many of these questions. I'll add problems to this sheet from time to time throughout the semester.

PROBLEMS

1 0-forms on 1-dimensional space

1. Give an interval $S = [a, b] \subset \mathbb{R}$ on which

$$\omega = |x - 4|$$

is a differential 0-form. Then give an interval $S' = [a', b'] \subset \mathbb{R}$ on which ω is not a differential 0-form.

2. Evaluate the integral

$$\int_{\partial S} 2x^2 + x$$

of the differential 0-form $\omega = 2x^2 + x$ over the boundary of the oriented interval $S = [3, -1]$.

3. Evaluate the integral

$$\int_{\partial S} x^3$$

of the differential 0-form $\omega = x^3$ over the boundary of $S = [2, 1] \cup [4, 3] \cup [-2, -1]$.

4. Is

$$\omega = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

a differential 0-form on the interval $S = [-1, 1]$? [See 4.4(b)]

5. Is the function

$$\omega = \begin{cases} x^2 \cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

differentiable at $x = 0$?

2 1-forms on 1-dimensional space

1. A state civic organization is conducting its annual fundraising campaign. Campaign expenditute will be incurred at a rate of \$10 000 per day. From past experience it is known that contrubutions will be high during the early stages of the campaign and will tend to fall off as the campaign continues. The rate at which contributions are received is modelled by the 1-form

$$w = (-100\,t^2 + 20\,000)\,dt\ .$$

What are the net proceeds expected to equal?

2. A hospital blood bank conducts an annual blood drive to replenish its inventory of blood. The hospital models the rate of blood donation by the 1-form

$$w = 300e^{-0.1t}\,dt$$

where t equals the length of the blood drive in days. If the goal for the blood drive is 2000 pints, when will the hospital reach its goal?

3. Find a differential 0-form ω whose derivative $d\omega$ is the differential 1-form

$$d\omega = (x^2 + 2x)\,dx\ .$$

4. Find a differential 0-form ω whose derivative $d\omega$ is the differential 1-form

$$d\omega = (x+2)\sin(x^2 + 4x - 6)\,dx\ .$$

[See 5.14(a)]

5. Find a differential 0-form ω whose derivative $d\omega$ is the differential 1-form

$$d\omega = \frac{6-x}{(x-3)(2x+5)}\,dx\ .$$

[See 5.20]

6. Find a differential 0-form ω whose derivative $d\omega$ is the differential 1-form

$$d\omega = \frac{1}{5 + 3\cos x}\,dx\ .$$

[See 5.21]

3 Fundamental theorem of calculus

1. Evaluate the integral
$$\int_S \frac{1}{\sqrt{(x+2)(3-x)}}\,dx$$
of the differential 1-form $\omega = dx/\sqrt{(x+2)(3-x)}$ over the oriented interval $S = [-1,1]$. [See 5.14(c)]

2. Evaluate the integral
$$\int_S \frac{1}{(x^2-2x+4)^{3/2}}\,dx$$
of the differential 1-form $\omega = (x^2-2x+4)^{-3/2}\,dx$ over the oriented interval $S = [2,1]$. [See 5.15]

3. Evaluate the integral
$$\int_S \frac{1}{x(\ln x)^3}\,dx$$
of the differential 1-form $\omega = dx/x(\ln x)^3$ over the oriented interval $S = [e, e^2]$. [See 5.16]

4. Give an informal proof of Stokes' formula $\int_{\partial S} \omega = \int_S d\omega$ for $S = [a,b] \subset \mathbb{R}$ and $\omega = f(x) \colon \mathbb{R} \to \mathbb{R}$ a differentiable function.

4 0-forms on n-dimensional space

1. For $v = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we define $||v|| = \sqrt{x_1^2 + \cdots + x_n^2}$. A function $f \colon \mathbb{R}^n \to \mathbb{R}^n$ is said to be *differentiable* at a point $c \in \mathbb{R}^n$ if there exists a linear function $T_c \colon \mathbb{R}^n \to \mathbb{R}^n$ such that
$$f(c+v) = f(c) + T_c(v) + ||v||E_c(v),$$
where $\lim_{v\to 0} E_c(v) = 0$.

   (a) Give an interpretation of the number $||v||$ for $n = 1, 2, 3$.

   (b) Give the definition of a limit such as $\lim_{v\to 0} E_c(v)$.

   (c) For $n = 1$ does this definition of differentiability agree precisely with the definition of differentiability given in the MA180 module?

   (d) For $n = 2$ give an informal interpretation of what it means for a function $f(x_1, x_2)$ of two variables to be differentiable at a point $c = (c_1, c_2) \in \mathbb{R}^2$.

   (e) Is the function $f(x_1, x_2) = |x_2|$ differentiable at the point $(1, 0) \in \mathbb{R}^2$?

2. Let S denote the oriented line segment in the plane going from the point $A = (1, 2)$ to the point $B = (-2, 3)$. Evaluate the integral

$$\int_{\partial S} x^2 + xy + y^2$$

of the differential 0-form $\omega = x^2 + xy + y^2$ over the boundary of S.

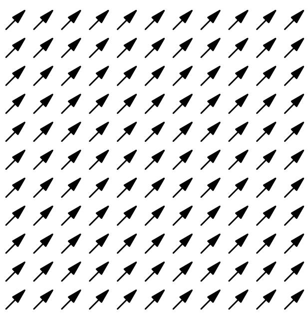
3. Let S denote the oriented line segment on the z-axis in $\mathbb{R}^3$ going from $z = 1$ to $z = 2$. Evaluate the integral

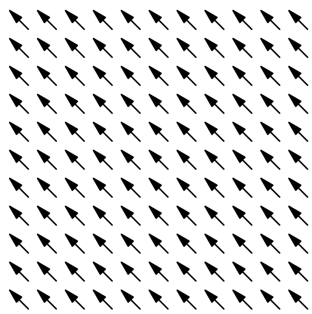
$$\int_{\partial S} z\, e^{x^2+y^2}$$

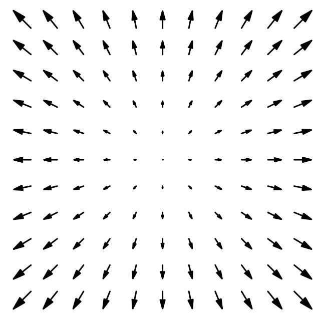
of the differential 0-form $\omega = z\, e^{x^2+y^2}$ over the boundary of S.

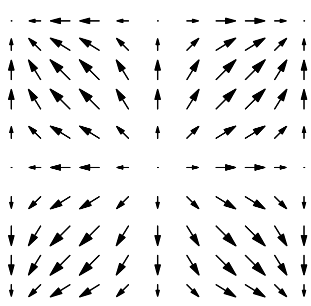
5 1-forms on n-dimensional space

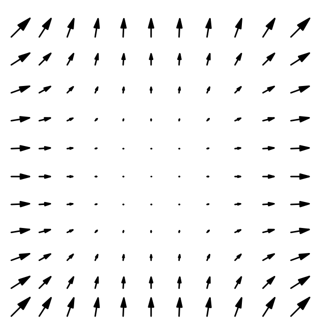
1. Match the five pictures of flows

(a) 

(b) 

(c) 

(d) 

(e) 

to the five differential 1-forms: (i) $\omega = x^2\, dx + y^2\, dy$, (ii) $\omega = \sin(\pi x)\, dx + \sin(\pi y)\, dy$, (iii) $\omega = x\, dx + y\, dy$, (iv) $\omega = dx + dy$, (v) $\omega = -dx + dy$.

2. In a constant force field the displacement of a particle

- from $(0, 0, 0)$ to $(4, 0, 0)$ needs 3 units of work;
- from $(1, -1, 0)$ to $(1, 1, 0)$ needs 2 units of work;
- from $(0, 0, 0)$ to $(3, 0, 2)$ needs 5 units of work.

Determine the differential 1-form that describes "work".

6 Integration of constant 1-forms

1. Evaluate the integral
$$\int_S 2\,dx + 3\,dy + 5\,dz$$
of the differential 1-form $\omega = 2\,dx + 3\,dy + 5\,dz$ on the line segment S in $\mathbb{R}^3$ starting at point $P = (3, 12, 4)$ and ending at point $Q = (11, 14, -7)$.

2. If work is given by the 1-form $3\,dx + 4\,dy - dz$ find all points which can be reached from the origin $(0, 0, 0)$ without work. Describe the set of these points geometrically.

7 Integration of 1-forms

1. Evaluate the integral
$$\int_S (x^2 - y)\,dx + (y^2 + x)\,dy$$
of the differential 1-form $\omega = (x^2 - y)\,dx + (y^2 + x)\,dy$ where $S \subset \mathbb{R}^2$ is the segment of the parabola $x = t$, $y = t^2 + 1$ from the point $(0, 1)$ to the point $(1, 2)$. [See 10.1]

2. An investment portfolio consists of two types of assets – type X and type Y. The marginal cost of varying the quantity of assets is modelled by the differential 1-form
$$w = k\,(x^2 - y)\,dx + k\,(y^2 + x)\,dy$$
where x is the volume of assets of type X, y is the volume of assets of type Y and k is some small constant. Currently $x = 0$ and $y = 100$. The managers wish to continuously restructure the portfolio so that $x = 100$ and $y = 200$. They ask three apprentice quants for their opinion. Apprentice A suggests that the volume of X should first be increased to 100, and after that the volume of Y should be increased to 200. Apprentice B suggests that the volume of both assets should be increased simulaneously, keeping the relationship $y = x+100$ throughout. Apprentice C suggests that the volume of both assets should be increased simulaneously, keeping the relationship $y = \frac{1}{100}x^2 + 100$ throughout. Which apprentice should be offered a permanent job?

3. Evaluate the integral
$$\int_C \omega$$
of the 1-form
$$\omega = (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz$$
where C is the straight line from $(0, 0, 0)$ to $(1, 1, 1)$. [See 10.2(c)]

4. Evaluate the integral
$$\int_C \omega$$
of the 1-form
$$\omega = (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz$$
where C is the curve $x = t$, $y = t^2$, $z = t^3$ from $(0,0,0)$ to $(1,1,1)$. [See 10.2(c)]

5. Evaluate the integral
$$\int_{\partial S} (2xy - x^2)\,dx + (x + y^2)\,dy$$
where ∂S is the boundary of the region S bounded by the two curves $y = x^2$ and $y^2 = x$. Assume an anti-clockwise orientation on ∂S. [See 10.6]

8 Differentiation of 0-forms

1. Determine the 1-form $d\omega$ arising as the derivative of the 0-form $\omega = x^2 e^{y/x}$. [See 6.16(a)]

2. Find a 0-form ω whose derivative is
$$d\omega = (3x^2y - 2y^2)\,dx + (x^3 - 4xy + 6y^2)\,dy.$$
[See 6.16(b)]

9 Partial derivatives

1. Suppose $U = z\sin(y/x)$ where $x = 3r^2 + 2s$, $y = 4r - 2s^3$ and $z = 2r^2 - 3^2$. Calculate $\partial u/\partial r$ and $\partial U/\partial s$. [See 6.22]

10 Fundamental Theorem of Calculus again

1. Evaluate
$$\int_S (6xy^2 - y^3)dx + (6x^2y - 3xy^2)dy\ .$$
where S is some path from $(1,2)$ to $(3,4)$. Explain why the integral is independent of the choice of path from $(1,2)$ to $(3,4)$. [See 10.14]

2. An investment portfolio consists of two types of assets – type X and type Y. The marginal cost of varying the quantity of assets is modelled by the differential 1-form

$$w = k(y^2\,dx + 2xy\,dy)$$

where x is the quantity of assets of type X, y is the quantity of assets of type Y, and k is a constant with value $k = 10^{-12}$. Currently $x = 100\,000$ and $y = 200\,000$. The managers wish to continuously restructure the portfolio so that $x = 250\,000$ and $y = 150\,000$. They wish to achieve the restructuring in a way that keeps the total cost to a minimum. How much should the total cost to be?

3. Evaluate

$$\int_S (2xy - y^4 + 3)dx + (x^2 - 4xy^3)dy\ .$$

where S is some path from $(1,0)$ to $(2,1)$. Explain why the integral is independent of the choice of path from $(1,0)$ to $(2,1)$. [See 10.48]

4. Prove that the differential 1-form

$$\omega = (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz$$

does not arise as the derivative $\omega = d\nu$ of any 0-form ν on $S - \mathbb{R}^3$. [See Questions 2 and 3 of Section 7 above]

5. Prove Stokes' formula $\int_{\partial S} \omega = \int_S d\omega$ for $\omega = f(x,y)\colon \mathbb{R}^2 \to \mathbb{R}$ a continuously differentiable function and $S \subset \mathbb{R}^2$ an oriented curve with differentiable parametrization $x = g(t)$, $y = h(t)$.

11 Constant 2-forms

1. Evaluate the integral

$$\int_S dx \wedge dy + 3dx \wedge dz$$

of the 2-form $\omega = dx \wedge dy + 3dx \wedge dz$ over the oriented triangle S with vertices $(0,0,0)$, $(1,2,3)$, $(1,4,0)$ in that order.

2. Evaluate the integral

$$\int_S dy \wedge dz + dz \wedge dx + dx \wedge dy$$

of the 2-form $\omega = dy \wedge dz + dz \wedge dx + dx \wedge dy$ over the oriented triangle S with vertices $(1,1,1)$, $(3,5,-1)$, $(4,2,1)$ in that order.

3. Evaluate the integral

$$\int_S 3dx \wedge dy$$

of the 2-form $\omega = 3dx \wedge dy$ over the region $S = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq 1\}$ where S is given a clockwise rotation when viewed from the positive z-axis.

The second in-class test is based on the above problems only.

12 More integration of 2-forms

1. Evaluate the integral

$$\int_S 3\, dx \wedge dy + 4\, dy \wedge dz$$

over the region $S = \{(x, y) \in \mathbb{R}^2 \ : \ x^2+y^2 \leq 1\}$ where S is given a clockwise rotation when viewed from the positive z-axis.

2. Let S be the region in the xy-plane bounded by $y = x^2$, $x = 2$ and $y = 1$. Let S have an anti-clockwise orientation. Evaluate

$$\int_S (x^2 + y^2 + z^2)\, dx \wedge dy.$$

[See 9.1]

3. Let S be the region in the xy-plane bounded by $y = x^2$, $x = 2$ and $y = 1$. Let S have an anti-clockwise orientation. Evaluate

$$\int_S (x^2 + y^2 + z^2)\, dy \wedge dz.$$

4. Let S be the region in the xy-plane bounded by the curves $y = x^2$, $y = \sqrt{2 - x^2}$, $x = 0$ and $x = 1$. Let S have an anti-clockwise orientation. Evaluate

$$\int_S xy\, dx \wedge dy.$$

[See 9.3(b)]

5. Find the volume of the region in $\mathbb{R}^3$ common to the intersecting cylinders $x^2+y^2 = 4$ and $x^2 + z^2 = 4$. [See 9.4]

6. A cinema customer queues X minutes for a ticket and Y minutes for popcorn. The probability densisty function for the pair (X, Y) is given by $f(x, y) = \frac{1}{50}e^{-10x}e^{-5y}$.

   (a) Express, as an integral of a differential 2-form over an oriented region S, the probability that a random customer waits less than 20 minutes in total.

   (b) Evaluate this integral.

13 Differentiation of k-forms

1. Find $d\omega$ for the following forms.

   (a) $\omega = xy\,dz + yz\,dx + zx\,dy$

   (b) $\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$

   (c) $\omega = e^{xyz}$

   (d) $\omega = (\cos x)\,dy + (\sin x)\,dz$

   (e) $\omega = (x + y)^2\,dy + (x + y)^2\,dz$

   (f) $\omega = \log x$

   (g) $\omega = x^2$

   (h) $\omega = \sin x$

   (i) $\omega = x$

2. Let $\omega = F(x, y, z)$ be a 0-form and assume $F_{xy} = F_{yx}$, $F_{xz} = F_{zx}$, $F_{yz} = F_{zy}$. Prove that $d(d\omega) = 0$.

3. Let $\omega = F(x, y, z)\,dx + G(x, y, z)\,dy + H(x, y, z)\,dz$ and assume that each of F, G, H satisfy the hypothesis of the preceding question. Prove that $d(d\omega) = 0$.

4. Use the preceding problem to prove that the differential 1-form

$$\omega = (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz$$

   does not arise as the derivative $\omega = d\nu$ of any 0-form ν on $S = \mathbb{R}^3$.

5. For two differential 0-forms ν, ω prove that

$$d(\nu\omega) = (d\nu)\omega + \nu(d\omega).$$

6. For two differential 1-forms $\nu = A\,dx + B\,dy$, $\omega = C\,dx + D\,dy$ prove that

$$d(\nu \wedge \omega) = (d\nu) \wedge \omega - \nu \wedge (d\omega).$$

14 Stokes' Formula

1. Verify Stokes' Formula $\int_{\partial S} \omega = \int_S d\omega$ for $\omega = (2xy - x^2)\,dx + (x + y^2)\,dy$ and S the region in the xy-plane bounded by $y = x^2$ and $x^2 = y$. [See 10.6]

2. Verify Stokes' Formula $\int_{\partial S} \omega = \int_S d\omega$ for $\omega = (2x-z)\,dy\wedge dz+x^2y\,dz\wedge dx-xz^2\,dx\wedge dy$ and S the region in $\mathbb{R}^3$ bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. (This is a very long and tedious questions to answer!) [See 10.23]

3. Let S denote the region bounded by some ellipse (or other some other simple closed curve) in the xy-plane. Use Stokes' Formula to show that the area of S is given by $\frac{1}{2}\int_{\partial S} x\,dy - y\,dx$. [See 10.8]

4. Calculate the area bounded by the ellipse $x = a\cos\theta$, $y = b\sin\theta$.

5. By considering an oriented 2-dimensional rectangle S in the xy-plane, explain how Stokes' formula $\int_{\partial S}\omega = \int_S d\omega$ leads to the definition of the derivative $d\omega$ of a differential 1-form $\omega = A\,dx + B\,dy$.

15 div, grad, curl

1. Consider the 0-form $\omega = (x^2 + y^2)/2$. Calculate the "gradient" 1-form $d\omega$ and sketch the corresponding vector field on $\mathbb{R}^2$.

2. Find a unit normal to the surface $S \subset \mathbb{R}^3$ defined by the equation

$$2x^2 + 4yz - 5z^2 = -10$$

at the pont $(3, -1, 2) \in S$. [See 7.37]

3. Find the equation of the tangent plane to the surface $S \subset \mathbb{R}^3$ defined by the equation

$$2x^2 + 4yz - 5z^2 = -10$$

at the pont $(3, -1, 2) \in S$.

4. Consider the 1-form $\omega = -y\,dx + x\,dy$. Sketch the corresponding vector field. Then compute the "curl" 2-form $d\omega$. What feature of your sketch is captured by $d\omega$?

5. Consider the vector field $F = xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k}$. Define curl($F$) in terms of the derivative of a 1-form and then calculate curl(F).

6. Consider the 0-form $\omega = x^2yz^3$ and the vector field $F = xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k}$. Determine grad($\omega$), div($F$), curl($F$), div($\omega F$), curl($\omega F$). [See 7.34]

16 First online homework

1. In the following mathematical text some words and symbols have been deleted and replaced by letters $\cdots\mathbf{A}\cdots$, $\cdots\mathbf{B}\cdots$ and so on. All of the removed words and symbols can be found in the list given after the mathematical text. Determine the list item for each letter.

START OF MATHEMATICAL TEXT

Recall that a function $T: \mathbb{R}^n \to \mathbb{R}^m$ $(m, n > 0)$ is a *linear transformation* if $T(a + \lambda b) = \cdots \mathbf{A} \cdots$ for $\lambda \in \mathbb{R}, a, b \in \cdots \mathbf{B} \cdots$.

Recall that a function $f: \mathbb{R} \to \mathbb{R}$ is *differentiable* at $a \in \mathbb{R}$ if there exists a number $\cdots \mathbf{C} \cdots \in \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \cdots \mathbf{D} \cdots . \tag{1}$$

If f is differentiable at a, then the function $D_a(h) = f'(a)h$ is a linear transformation $D_a: \mathbb{R} \to \mathbb{R}, h \mapsto D_a(h)$ and equation (1) is equivalent to

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - D_a(h)}{h} = \cdots \mathbf{E} \cdots . \tag{2}$$

.

The definition of differentiability can be reformulated as follows. One can define a function $f: \mathbb{R} \to \mathbb{R}$ to be *differentiable* at $a \in \mathbb{R}$ if there exists a linear transformation $D_a: \mathbb{R} \to \mathbb{R}$ such that equation $\cdots \mathbf{F} \cdots$ holds. More generally, a function $f: \mathbb{R}^n \to \mathbb{R}^m$ is *differentiable* at $a \in \mathbb{R}^n$ if there is a linear transformation $D_a: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{||f(a+h) - f(a) - D_a(h)||}{||h||} = \cdots \mathbf{G} \cdots \tag{3}$$

where h varies in $\cdots \mathbf{H} \cdots$. Here the *norm* of $x = (x_1, \ldots, x_k)$ is defined as $||x|| = \cdots \mathbf{I} \cdots$.

The transformation $D_a: \mathbb{R}^n \to \mathbb{R}^m$ is called *the derivative* of f at a. The use of the $\cdots \mathbf{J} \cdots$ article is justified by the following theorem.

Theorem. If $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$ there is a *unique* linear transformation $D_a: \mathbb{R}^n \to \mathbb{R}^m$ for which $\cdots \mathbf{K} \cdots$ holds.

Proof. Suppose that f is differentiable at a with derivative D_a satisfying $\cdots \mathbf{L} \cdots$. Suppose that some linear transformation $\cdots \mathbf{M} \cdots$ satisfies

$$\lim_{h \to 0} \frac{||f(a+h) - f(a) - T_a(h)||}{||h||} = \cdots \mathbf{N} \cdots . \tag{4}$$

For $h \in \mathbb{R}^n$ define $\mu(h) = f(a+h) - f(a)$. Then

$$\begin{aligned} \lim_{h \to 0} \tfrac{||D_a(h) - T_a(h)||}{||h||} &= \lim_{h \to 0} \tfrac{||D_a(h) - \mu(h) + \mu(h) - T_a(h)||}{||h||} \\ &\cdots \mathbf{O} \cdots \lim_{h \to 0} \tfrac{||D_a(h) - \mu(h)||}{||h||} + \lim_{h \to 0} \tfrac{||\mu(h) - T_a(h)||}{||h||} \\ &= \cdots \mathbf{P} \cdots . \end{aligned} \tag{5}$$

If $x \in \mathbb{R}^n$ then $tx \to 0$ as $\cdots$ **Q** $\cdots$. Hence for $x \neq 0$ we have

$$
\begin{aligned}
\cdots \mathbf{R} \cdots &= \lim_{t\to 0} \frac{||D_a(tx)-T_a(tx)||}{||tx||} \\
&= \lim_{t\to 0} \frac{||tD_a(x)-tT_a(x)||}{||tx||} \\
&= \lim_{t\to 0} \frac{t||D_a(x)-T_a(x)||}{t||x||} \\
&= \frac{||D_a(x)-T_a(x)||}{||x||}.
\end{aligned} \tag{6}
$$

Hence $\cdots$ **S** $\cdots$. **Q.E.D.**

END OF MATHEMATICAL TEXT

LIST OF WORDS AND SYMBOLS

(a) $T_a\colon \mathbb{R}^n \to \mathbb{R}^m$

(b) $\sqrt{x_1^2 + x_2^2 + \cdots + x_k^2}$

(c) $\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

(d) $t \to 0$

(e) 0

(f) (0)

(g) 1

(h) (1)

(i) indefinite

(j) determinate

(k) indeterminate

(l) definite

(m) $D_a(x) = T_a(x)$

(n) f'

(o) $f'(x)$

(p) $f'(a)$

(q) (2)

(r) $\leq$

(s) $=$

(t) $\geq$

(u) (3)

(v) (4)

(w) $T(a) + T(b)$

(x) $T(a) + T(\lambda b)$

(y) $T(a) + \lambda T(b)$

(z) $\mathbb{R}^n$

(aa) $\mathbb{R}^m$

(ab) Done

2. In the preceding proof, which (if any) of the equations (4), (5), (6) in some way uses the linearity of D_a or T_a.

3. Let $D_a\colon \mathbb{R}^n \to \mathbb{R}^m$ be the derivative at $a \in \mathbb{R}^n$ of some function $f\colon \mathbb{R}^n \to \mathbb{R}^m$. It is often convenient to consider the matrix of this D_a with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. This $m \times n$ matrix is called the *Jacobian* of f at a.

   Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (\sin(x), \cos(y))$. Its Jacobian at the point $(x, y) \in \mathbb{R}^2$ will be a 2×2 matrix

$$\begin{pmatrix} s & t \\ u & v \end{pmatrix} .$$

   Determine this matrix.

17 Second online homework

Read the following mathematical text, and then attempt the questions which are designed to assess your understanding of it.

MATHEMATICAL TEXT

Let $\mathbb{R}^n$ denote the set of all *vectors* $v = (v_1, \ldots, v_n)$ with $v_1, \ldots, v_n \in \mathbb{R}$. Two vectors $v = (v_1, \ldots, v_n)$, $v' = (v'_1, \ldots, v'_n)$ can be added componentwise:

$$v + v' = (v_1 + v'_1, \cdots, v_n + v'_n) .$$

The vector v can be multiplied by a *scalar* $\lambda \in \mathbb{R}$ componentwise:

$$\lambda v = (\lambda v_1, \ldots, \lambda v_n) .$$

A function $\phi\colon \mathbb{R}^n \to \mathbb{R}^m, v \mapsto \phi(v)$ is a *linear homomorphism* if

$$\phi(v + \lambda v') = \phi(v) + \lambda \phi(v')$$

for all $v, v' \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

A function $\phi: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m, (u, v) \mapsto \phi(u, v)$ is an *anti-symmetric bilinear homomorphism* if

$$\phi(u, v + \lambda v') = \phi(u, v) + \lambda\phi(u, v') \ ,$$
$$\phi(u + \lambda u', v) = \phi(u, v) + \lambda\phi(u', v) \ ,$$
$$\phi(u, v) = -\phi(v, u)$$

for $u, u', v, v' \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

A 0-*form* on a set $D \subset \mathbb{R}^n$ is just another name for a real valued function

$$\omega: D \to \mathbb{R}, x \mapsto \omega(x) \ .$$

A 1-*form* on a set $D \subset \mathbb{R}^n$ is a real valued function

$$\omega: D \times \mathbb{R}^n \to \mathbb{R}, (x, v) \mapsto \omega(x, v)$$

such that for each $x \in D$ the function

$$\mathbb{R}^n \to \mathbb{R}, v \mapsto \omega(x, v)$$

is a linear homomorphism.

A 2-*form* on a set $D \subset \mathbb{R}^n$ is a real valued function

$$\omega: D \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, (x, u, v) \mapsto \omega(x, u, v)$$

such that for each $x \in D$ the function

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, (u, v) \mapsto \omega(x, u, v)$$

is an anti-symmetric bilinear homomorphism.

Let $\omega = \omega(x, u)$ and $\nu = \nu(x, v)$ be two 1-forms on a set $D \subset \mathbb{R}^n$. Their *wedge product* $\omega \wedge \nu$ is the 2-form

$$\omega \wedge \nu: D \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, (x, u, v) \mapsto \begin{vmatrix} \omega(x, u) & \omega(x, v) \\ \nu(x, u) & \nu(x, v) \end{vmatrix}$$

involving the determinant of a 2×2 matrix.

Remark: We use the term *differential k-form* to mean a *k-form* that satisfies certain differentiability criteria.

END OF MATHEMATICAL TEXT

1. Suppose that ω is a 1-form on $\mathbb{R}^2$ satisfying $\omega(x,(1,0)) = 5$ and $\omega(x,(0,1)) = 3$ for all $x \in \mathbb{R}^2$.

   Determine $\omega(x,(2,4))$.

2. Suppose that ω is a 1-form on $\mathbb{R}^2$ satisfying $\omega(x,(1,1)) = 5$ and $\omega(x,(2,1)) = 3$ for all $x \in \mathbb{R}^2$. Determine $\omega(x,(7,5))$.

3. Suppose that ω is a 2-form on $\mathbb{R}^2$ satisfying

$$\omega(x,(1,0),(1,0)) = 1,$$

$$\omega(x,(0,1),(1,0)) = 2,$$
$$\omega(x,(1,0),(0,1)) = -3,$$
$$\omega(x,(0,1),(0,1)) = 1,$$

   for all $x \in \mathbb{R}^2$.

   Determine $\omega(x,(2,4),(1,3))$.

4. Suppose that $\omega(x,u,v)$ is a 2-form on a set $D \subset \mathbb{R}^2$.

   Determine $\omega(x,(2,4),(2,4))$.

5. Consider the differential 1-forms

$$\omega = xy\,dx - y\,dy \ ,$$

$$\nu = x^2\,dx + y^2\,dy$$

   on the set $D = \mathbb{R}^2$. Here were view ω as a function $\omega\colon D \times \mathbb{R}^2 \to \mathbb{R}, (x,(dx,dy)) \mapsto xy\,dx - y\,dy$. We view ν analogously.

   Evaluate the differential 2-form

$$\omega \wedge \nu\colon D \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, (x,u,v) \mapsto \omega \wedge \nu(x,u,v)$$

   for $u = (2,3)$, $v = (2,3)$ and arbitrary $x \in D$.