2.5 Elementary Row Operations and the Determinant

Recall: Let A be a 2×2 matrtix : $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The *determinant* of A, denoted by $\det(A)$ or $|A|$, is the number $ad - bc$. So for example if

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix}, \ \det(A) = 2(5) - 4(1) = 6.$$

The matrix A is invertible if and only if $\det(A) \neq 0$, and in this case the inverse of A is given by

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The matrix $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is called the *adjoint* or *adjugate* of A, denoted $\mathrm{adj}(A)$.

Determinants are defined for all square matrices. They have various interpretations and applications in algebra, analysis and geometry. For every square matrix A, we have that A is invertible if and only if $\det(A) \neq 0$.

If A is a 2×2 matrix, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $|\det(A)|$ is the volume of the parallelogram having the vectors $\vec{v}_1 = (a, b)$ and $\vec{v}_2 = (c, d)$ as edges. Similarly if

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

is a 3×3 matrix we have $|\det(A)| =$ Volume of P, where P is the parallelepiped in $\mathbb{R}^3$ having $\vec{v}_1 = (a, b, c)$, $\vec{v}_2 = (d, e, f)$ and $\vec{v}_3 = (g, h, i)$ as edges.

The determinant of an $n \times n$ matrix can be defined recursively in terms of determinants of $(n-1) \times (n-1)$ matrices (which in turn are defined in terms of $(n-2) \times (n-2)$ determinants, etc.).

Definition 2.5.1 *Let A be a $n \times n$ matrix. For each entry $(A)_{ij}$ of A, we define the* minor *M_{ij} of $(A)_{ij}$ to be the determinant of the $(n-1) \times (n-1)$ matrix which remains when the ith row and jth column (i.e. the row and column containing $(A)_{ij}$) are deleted from A.*

Example: Let $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & -2 & 1 \\ -4 & 1 & -1 \end{pmatrix}$.

$$M_{11} : M_{11} = \det \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} = -2(-1) - (1)(1) = 1$$

$$M_{12} : M_{12} = \det \begin{pmatrix} 2 & 1 \\ -4 & -1 \end{pmatrix} = 2(-1) - (1)(-4) = 2$$

$$M_{22} : M_{22} = \det \begin{pmatrix} 1 & 0 \\ -4 & -1 \end{pmatrix} = 1(-1) - (0)(-4) = -1$$

$$M_{23} : M_{23} = \det \begin{pmatrix} 1 & 3 \\ -4 & 1 \end{pmatrix} = 1(1) - (3)(-4) = 13$$

$$M_{32} : M_{32} = \det \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = 1(1) - (0)(2) = 1$$

Definition 2.5.2 *We define the* cofactor C_{ij} *of the entry* $(A)_{ij}$ *of* A *as follows:*

$$\begin{aligned} C_{ij} &= M_{ij} \quad \textit{if} \quad i+j \textit{ is even} \\ C_{ij} &= -M_{ij} \quad \textit{if} \quad i+j \textit{ is odd} \end{aligned}$$

So the cofactor C_{ij} is either equal to $+M_{ij}$ or $-M_{ij}$, depending on the position (i, j).

We have the following pattern of signs : in the positions marked "−", $C_{ij} = -M_{ij}$, and in the positions marked "+", $C_{ij} = M_{ij}$:

$$\begin{pmatrix} + & - & + & + & \dots \\ - & + & - & \dots & \\ + & - & + & \dots & \\ - & + & \dots & & \\ \vdots & & & & \end{pmatrix} \text{ e.g. for } 3 \times 3 \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

<u>Example</u>: $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & -2 & 1 \\ -4 & 1 & -1 \end{pmatrix}$

$$C_{11} : C_11 = M_{11} = \det \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} = -2(-1) - (1)(1) = 1$$

$$C_{12} : C_{12} = -M_{12} = -\det \begin{pmatrix} 2 & 1 \\ -4 & -1 \end{pmatrix} = -(2(-1) - (1)(-4)) = -2$$

$$C_{22} : C_{22} = M_{22} = \det\begin{pmatrix} 1 & 0 \\ -4 & -1 \end{pmatrix} = 1(-1) - (0)(-4) = -1$$

$$C_{23} : C_{23} = -M_{23} = -\det\begin{pmatrix} 1 & 3 \\ -4 & 1 \end{pmatrix} = -(1(1) - (3)(-4)) = -13$$

$$C_{32} : C_{32} = -M_{32} = -\det\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = -(1(1) - (0)(2)) = -1$$

Definition 2.5.3 *The* determinant $\det(A)$ *of the $n \times n$ matrix A is calculated as follows :*

1. *Choose a row or column of A.*
2. *For every A_{ij} in the chosen row or column, calculate its cofactor.*
3. *Multiply each entry of the chosen row or column by its own cofactor.*
4. *The sum of these products is* $\det(A)$.

Examples:

1. Let $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -2 & 2 & 3 \end{pmatrix}$. Find $\det(A)$.

   Solution: We can calculate the determinant using cofactor expansion along the first row.

   Find the cofactors of the entries in the 1st row of A:

$$\begin{aligned} C_{11} &= +\det\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = 4 \\ C_{12} &= -\det\begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} = 1 \\ C_{13} &= +\det\begin{pmatrix} -1 & 2 \\ -2 & 2 \end{pmatrix} = 2 \end{aligned}$$

   Then

$$\begin{aligned} &= A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13} \\ &= 2(4) + 1(1) + 3(2) \\ &= 15 \end{aligned}$$

Note: We could also do the cofactor expansion along the 2nd row:

$$\det(A) = \underbrace{A_{21}C_{21} + A_{22}C_{22} + A_{23}C_{23}}_{\text{entries of 2nd row of } A \text{ multiplied by their cofactors}}$$

$$\begin{aligned} C_{21} &= -\det\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} = 3 \\ C_{22} &= +\det\begin{pmatrix} 2 & 3 \\ -2 & 3 \end{pmatrix} = 12 \\ C_{23} &= -\det\begin{pmatrix} 2 & 1 \\ -2 & 2 \end{pmatrix} = -6 \end{aligned}$$

$$\det(A) = -1(3) + 2(12) + 1(-6) = 15$$

2. Let $B = \begin{pmatrix} 3 & 1 & 5 & -24 \\ 0 & 4 & 1 & -6 \\ 0 & 0 & 25 & 4 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. Calculate $\det(A)$.

Solution: Use cofactor expansion along the first column to obtain :

$$\det(B) = 3\det\begin{pmatrix} 4 & 1 & -6 \\ 0 & 25 & 4 \\ 0 & 0 & -1 \end{pmatrix}$$

On this 3×3 determinant, use the first column again. Then

$$\det(B) = 3 \times 4 \det\begin{pmatrix} 25 & 4 \\ 0 & -1 \end{pmatrix}$$

On this 2×2 determinant, use the first column again. Then

$$\det(B) = 3 \times 4 \times 25 \times -1 = -300.$$

Notes

1. The matrix B above is an example of an *upper triangular matrix* (all of its non-zero entries are located on or above its main diagonal). Note that $\det(B)$ is just the product of the entries along the main diagonal of B.

2. If calculating a determinant using cofactor expansion, it is usually a good idea to choose a row or column containing as many zeroes as possible.

Definition: A $n \times n$ matrix A is called *upper triangular* if all entries located below (and to the left of) its main diagonal are zeroes (i.e. if $A_{ij} = 0$ whenever $i > j$).

(In the following diagram the entries indicated by "$*$" may be any real number.)

$$\begin{pmatrix} * & * & \dots & \dots & * \\ 0 & * & * & \dots & * \\ 0 & 0 & * & \vdots & \vdots \\ \vdots & \vdots & 0 & \ddots & * \\ 0 & \dots & \dots & 0 & * \end{pmatrix}$$

Upper triangular matrix

Theorem 2.5.4 *If A is upper triangular, then* $\det A$ *is the product of the entries on the main diagonal of A.*

The idea of the proof of Theorem 2.5.4 is suggested by Example 2 above - just use cofactor expansion along the first column.

Consequence of Theorem 2.5.4: An upper triangular matrix is invertible if and only if none of the entries along its main diagonal is zero.

So determinants of upper triangular matrices are particularly easy to calculate. This fact can be used to calculate the determinant of any square matrix, after using elementary row operations to reduce it to row echelon form.

The following table describes the effect on the determinant of a square matrix of ERO's of the three types.

	Type of ERO	Effect on Determinant
1.	Add a multiple of one row to another row	No effect
2.	Multiply a row by a constant c	Determinant is multiplied by c
3.	Interchange two rows	Determinant changes sign

We can use these facts to find the determinant of any $n \times n$ matrix A as follows :

1. Use elementary row operations (ERO's) to obtain an upper triangular matrix A' from A.

2. Find $\det A'$ (product of entries on main diagonal).

3. Make adjustments to reverse changes to the determinant caused by ERO's in Step 1.

Example 2.5.5 *Find the determinant of the matrix*

$$A = \begin{pmatrix} 2 & 4 & 2 & 1 \\ 4 & 3 & 0 & -1 \\ -6 & 0 & 2 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

<u>Solution</u>:

Step 1: Perform elementary row operations to reduce A to upper triangular form.

$$\begin{pmatrix} 2 & 4 & 2 & 1 \\ 4 & 3 & 0 & -1 \\ -6 & 0 & 2 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix} \underset{R_3+3R_1}{\overset{R_2-2R_1}{\longrightarrow}} \begin{pmatrix} 2 & 4 & 2 & 1 \\ 0 & -5 & -4 & -3 \\ 0 & 12 & 8 & 3 \\ 0 & 1 & 1 & 2 \end{pmatrix} \underset{(\det\times(-1))}{\overset{R_2 \leftrightarrow R_4}{\longrightarrow}}$$

$$\begin{pmatrix} 2 & 4 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 12 & 8 & 3 \\ 0 & -5 & -4 & -3 \end{pmatrix} \underset{R_4+5R_2}{\overset{R_3-12R_2}{\longrightarrow}} \begin{pmatrix} 2 & 4 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -21 \\ 0 & 0 & 1 & 7 \end{pmatrix} \underset{(\det\times(-1))}{\overset{R_3 \leftrightarrow R_4}{\longrightarrow}}$$

$$\begin{pmatrix} 2 & 4 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & -4 & -21 \end{pmatrix} \overset{R_4+4R_3}{\longrightarrow} \begin{pmatrix} 2 & 4 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

Step 2: Call this upper triangular matrix A'. Then $\det A' = 2 \times 1 \times 1 \times 7 = 14$.

Step 3: $\det(A') = \det(A)$ since the determinant changed sign twice during the row reduction at Step 1 but was otherwise unchanged. Thus

$$\det(A) = \det(A') = 2 \times 1 \times 1 \times 7 = 14$$

Explanation of Effects of EROs on the Determinant

	Type of ERO	Effect on Determinant
1.	Multiply a row by a constant c	Determinant is multiplied by c
2.	Add a multiple of one row to another row	No effect
3.	Interchange two rows	Determinant changes sign

1. Suppose that a square matrix A' results from multiplying Row i of A by the non-zero constant c. Using cofactor expansion by Row i to calculate $\det(A)$ and $\det(A')$, and using C_{ij} and C'_{ij} to denote the cofactors of the entries in the (i, j)-positions of A and A' resepctively, we find

$$\begin{aligned}\det(A) &= A_{i1}C_{i1} + A_{i2}C_{i2} + \cdots + A_{in}C_{in} \\ \det(A') &= A'_{i1}C'_{i1} + A'_{i2}C'_{i2} + \cdots + A'_{in}C'_{in} \\ &= cA_{i1}C'_{i1} + cA_{i2}C'_{i2} + \cdots + cA_{in}C'_{in} \\ &= c(A_{i1}C'_{i1} + A_{i2}C'_{i2} + \cdots + A_{in}C'_{in}).\end{aligned}$$

Since A and A' have the same entries outside Row i, the cofactors of entries in Row i of A and Row i of A' are the same. Thus

$$\det(A') = c(A_{i1}C_{i1} + A_{i2}C_{i2} + \cdots + A_{in}C_{in}) = c\det(A).$$

2. First suppose that in a square matrix B, Row 2 is a multiple of Row 1. This means that there is a real number c for which

$$B_{21} = cB_{11},\ B_{22} = cB_{12}, \dots, B_{2n} = cB_{in}.$$

If we subtract $c\times$Row i from Row k of B, we obtain a matrix having a full row of zeroes. Thus the RREF obtainable from B by EROs is not the $n\times n$ identity matrix, as it contains at least one row full of zeroes. Hence B is not invertible and $\det(B) = 0$.

Thus : any matrix in which one row is a multiple of another has determinant zero.

Now suppose that A' is obtained from A by adding $c\times$Row k to Row i. So the entries in Row i of A' are

$$cA_{k_1} + A_{i1},\ cA_{k2} + A_{i2},\ \dots, cA_{kn} + A_{in}$$

Outside Row i, A and A' have the same entries. Hence the cofactors of the entries in Row i of A and A' are the same. We let C_{ij} denote the cofactor of the entry in the (i, j) position of either of these matrices. Now if we calculate $\det(A)$ and $\det(A')$ using Row i we obtain

$$\begin{aligned}
\det(A) &= A_{i1}C_{i1} + A_{i2}C_{i2} + \cdots + A_{in}C_{in} \\
\det(A') &= (cA_{k_1} + A_{i1})C_{i1} + (cA_{k2} + A_{i2})C_{i2} + \cdots + (cA_{kn} + A_{in})C_{in} \\
&= (cA_{k1}C_{i1} + cA_{k2}C_{i2} + \cdots + cA_{in}C_{in}) + (A_{i1}C_{i1} + A_{i2}C_{i2} + \cdots + A_{in}C_{in})
\end{aligned}$$

Now

$$cA_{k1}C_{i1} + cA_{k2}C_{i2} + \cdots + cA_{in}C_{in}$$

is the determinant of a matrix in which Row k is just $c\times$Row i, hence this number is zero by the remarks above. Thus

$$\det(A') = A_{i1}C_{i1} + A_{i2}C_{i2} + \cdots + A_{in}C_{in} = \det(A).$$

3. We omit the full details, but consider the case where A' is obtained from A by swapping the first two rows. Use M_{ij} and M'_{ij} to denote minors of A and A', and C_{ij} and C'_{ij} respectively for cofactors of A and A'. Using the first row of A and the second row of A' we obtain

$$\begin{aligned}
\det(A) &= A_{11}C_{11} + A_{12}C_{12} + \cdots + A_{1n}C_{1n} \\
&= A_{11}M_{11} - A_{12}M_{12} + \cdots \pm A_{1n}C_{1n}. \\
\det(A') &= A'_{21}C'_{21} + A'_{22}C'_{22} + \cdots + A'_{2n}C'_{2n} \\
&= A_{11}C'_{21} + A_{12}C'_{22} + \cdots + A_{1n}C'_{2n} \\
&= -A_{11}M'_{21} + A_{12}M'_{22} - \cdots \pm A_{1n}M'_{1n} \\
&= -A_{11}M_{11} + A_{12}M_{12} + \cdots \pm A_{1n}M_{1n} \\
&= -\det(A).
\end{aligned}$$

The case of general row swaps is messier but basically similar.