Chapter 3

The Laplacian Matrix of a Graph

3.1 Introduction to the graph Laplacian

Definition 3.1.1. *Let* G *be a graph. The* Laplacian matrix *of* G*, denoted* $L(G)$*, is defined by* $L(G) = \Delta(G) - A(G)$*, where* $A(G)$ *is the adjacency matrix of* G *and* $\Delta(G)$ *is the diagonal matrix whose* (i,i) *entry is equal to the degree of the* i*th vertex of* G*.*

The Laplacian matrix of a graph carries the same information as the adjacency matrix obviously, but has different useful and important properties, many relating to its spectrum. We start with a few examples.
Examples

1. *Complete graphs* If $G = K_4$ then $L(G) = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$. We can observe that $v_1 = (1\ 1\ 1\ 1)^T$ is an eigenvector of $L(G)$ corresponding to the eigenvalue 0, since the row sums in $L(G)$ are all equal to zero. This is true of the Laplacian matrix of any graph, and it follows from the fact that in each row we have the degree of the corresponding vertex on the diagonal, along with a -1 for each of its incident edges. At this point we don't know the multiplicity of the zero eigenvalue, but we know from Theorem 2.2.2 that any eigenvector corresponding to a non-zero eigenvalue must be orthogonal to v_1, which means that the sum of its entries must be zero. So suppose that $a+b+c+d=0$ (with a,b,c,d not all zero) and consider the equation

$$\begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

with $\lambda \neq 0$. This says

$$\begin{aligned} (3-\lambda)a &= b+c+d &\implies& \quad (3-\lambda)a = -a \\ (3-\lambda)b &= a+c+d &\implies& \quad (3-\lambda)b = -b \\ (3-\lambda)c &= a+b+d &\implies& \quad (3-\lambda)c = -c \\ (3-\lambda)d &= a+b+c &\implies& \quad (3-\lambda)d = -d \end{aligned}$$

Any choice of a,b,c,d with $a+b+c+d=0$ satisfies these equations, with $3-\lambda = -1$, so $\lambda = 4$. The 3-dimensional subspace $\langle v_1 \rangle^{\perp}$ of $\mathbb{R}^4$ consists entirely of eigenvectors of $L(G)$ corresponding to the eigenvalue 4, so this eigenvalue occurs with multiplicity 3 and

$\operatorname{specL}(G) = [0, 4, 4, 4]$. Note that the sum of the eigenvalues is 3×4 which is also the trace as expected.

In general, $\operatorname{specL}(K_n) = [0, \underbrace{n, \ldots, n}_{n-1}]$.

For the complete graphs, all the non-zero eigenvalues coincide. The greatest is n which is also the graph order.

2. *Cycles*

Let C_4 be the cycle of length 4. Then $L(G) = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$.

As above, an eigenvector of $L(G)$ corresponding to a non-zero eigenvalue λ is a non-zero vector whose entries sum to zero and satisfy

$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

This means

$$\begin{aligned} (2-\lambda)a = b + d &\implies (2-\lambda)a = b + d \\ (2-\lambda)b = a + c &\implies (2-\lambda)b = a + c \\ (2-\lambda)c = b + d &\implies (2-\lambda)c = b + d \\ (2-\lambda)d = a + c &\implies (2-\lambda)d = a + c \end{aligned}$$

Adding the first two equations gives $(2-\lambda)(a+b) = 0$, which means that $\lambda = 2$ or $a+b = 0$. The possibility that $\lambda = 2$ gives two independent (and orthogonal) eigenvectors

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \; v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

The remaining possibility $a + b = 0$ gives $c + d = 0$ also. Putting $a = 1$ we find the eigenvector

$$v_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

corresponding to the eigenvalue 4. So the Laplacian spectrum of C_4 is $[0, 2, 2, 4]$. Again the greatest eigenvalue is 4 (equal to the order) and the least positive eigenvalue is 2 this time.

In general the Laplacian spectrum of C_n is $[2 - 2\cos(\frac{2\pi k}{n}), k = 0 \ldots n-1]$. All eigenvalues are in the range 0 to 4, and the least positive eigenvalue approaches 0 as n increases, and occurs with multilicity 2. The greatest eigenvalue is 4 exactly if n is even (when π is an integer multiple of $\frac{2\pi k}{n}$).

3. *Stars* Let $G = S_n$, the star on n vertices. This graph has one vertex that is adjacent to all others, which have degree 1. We take $n = 4$ as an example. Then

$$L(G) = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

As above we consider

$$\begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

with $a + b + c + d = 0$ and $\lambda \neq 0$. Then

$$\begin{aligned} (3-\lambda)a = b + c + d \quad &\implies \quad (3-\lambda)a = -a \\ (1-\lambda)b = a & \\ (1-\lambda)c = a & \\ (1-\lambda)d = a & \end{aligned}$$

If $a \neq 0$, then $3 - \lambda = -1$ and $\lambda = 4$. We then find $-b = -c = -d = a$, so we obtain the eigenvector

$$\begin{pmatrix} 3 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

corresponding to the eigenvalue 4.
Alternatively if $a = 0$ then remaining eigenvectors are in the 2-dimensional space of vectors satisfying $b + c + d = 0$. We find that all elements of this space are eigenvectors of $L(S_4)$ corresponding to $\lambda = 1$. So the spectrum of $L(S_4)$ is $[0, 1, 1, 4]$. The greatest eigenvalue is 4 again, and the least positive eigenvalue is 1, which occurs twice.

In general the Laplacian spectrum of S_n is $[0, \underbrace{1, \dots, 1}_{n-2}, n]$ (this is not too hard to check). The minimum positive eigenvalue is 1 this time, and it occurs with multiplicity $n - 2$.

Theorem 3.1.2. *The Laplacian matrix of a graph G is a positive semidefinite matrix.*

Proof. Let B be the incidence matrix of G, in which rows are labelled by the edges of G, columns by the vertices of G, and the entry in the (i, j) position is 1 or 0 according to whether vertex j is incident with edge i or not. Thus each row of B has exactly two 1s, and a the number of 1s in Column j of B is the degree of vertex j. Now adjust B by changing the first 1 in each row to -1 and leaving all other entries alone. (Now B_1 is what is called an "oriented incidence matrix" for G, writing the two non-zero entries in Row i as 1 and -1 can be interpreted as assigning a direction to edge i).

The square matrix $B_1^T B_1$ has rows and columns labelled by the vertices $v_1, \dots, v_n$ of G. Its entry in the (i, j) position is the scalar product of Columns i and j of B_1. This is $\deg(v_i)$ if $i = j$, and if $i \neq j$ it is 0 *unless* there is a row in which both Column i and Column j have nonzero entries. There can be at most one such row and it occurs when $v_i v_j$ is an edge of G, in which case the scalar product of Columns i and j of B_1 is $(1)(-1) = -1$. Thus

$$(B_1^T B_1)_{ij} = \begin{cases} \deg(v_i) & \text{if} \quad i = j \\ 0 & \text{if} \quad i \not\sim j \\ -1 & \text{if} \quad i \sim j \end{cases}$$

Thus $B_1^T B_1 = L(G)$ and $L(G)$ is positive semidefinite by Lemma 2.1.8. □

Thus all eigenvalues of the Laplacian matrix of a graph are non-negative, and the zero eigenvalue occurs with multiplicy at least 1, since the row sums are all zero. Our next main result is that the multiplicity of the zero eigenvalue tells us the number of connected components.

Theorem 3.1.3. *Let G be a graph. Then the dimension of the nullspace of $L(G)$ is the number of connected components of G.*

Since the matrix $L(G)$ is symmetric and therefore diagonalizable, the multiplicity of zero as a root of its characteristic polynomial is the same as the dimension of the right nullspace of $L(G)$ which is the geometric multiplicity of zero as an eigenvalue of $L(G)$. So in order to prove Theorem 3.1.3 it is enough to consider the right nullspace of $L(G)$.

Proof. Let B be an oriented incidence matrix of G and write $L(G) = B^T B$, where $L(G)$ and B are written with respect to the ordering $v_1, \dots, v_n$ of the vertices of G. Suppose that $x \in \mathbb{R}^n$ is an eigenvector of $L(G)$ corresponding to 0, i.e. that $L(G)x = 0$. Then

$$B^T B x = 0 \implies x^T B^T B x = 0 \implies (Bx)^T Bx = 0.$$

Thus Bx is a self-orthogonal vector in $\mathbb{R}^n$ which means $Bx = 0$, and it is enough to consider the right nullspace of B. If the column vector x is orthogonal to every row of B, it means that the components in the $x_i = x_j$ whenver the vertices v_i and v_j are adjacent in G. Thus x_i and x_j must be equal whenever there is a path from v_i to v_j in G, i.e. whenever v_i and v_j belong to the same component of G.

Let $C_1, \dots, C_k$ be the connected components of G, and for $i \in \{1, \dots, k\}$ let u_i be the vector that has 1s in the positions corresponding to the vertices of C_i and zeros elsewhere. Then u_i is easily confirmed to be in the right nullspace of $L(G)$, and by the above argument every element of this nullspace is a linear combination of $u_1, \dots, u_k$. Since these vectors are linearly independent, they form a basis of the zero eigenspace of $L(G)$, and the dimension of this space is k, the number of components. □

An alternative version of Theorem 3.1.3 expresses the rank of $L(G)$ as $n-$(the number of components). Thus the graph Laplacian provides a feasible means for determining the number of components in a graph. There is no direct way of reading this number from the adjacency matrix.

Just as the multiplicity of the zero eigenvalue of $L(G)$ carries information about the number of connected components of G, the appearance an/or multiplicity of the eigenvalue n tells us about components of the complement $\overline{G}$.

Theorem 3.1.4. *Suppose that G is a graph of order n and that n occurs c times as an eigenvalue of $L(G)$, where $c \geqslant 0$. Then the number of connected components of $\overline{G}$ is $c+1$.*

Example We have seen that n occurs $n-1$ times as an eigenvalue of $L(K_n)$. The complement of K_n has n isolated vertices and so has n connected components. The star S_n has n appearing once as an eigenvalue, and its complement has two components - an isolated vertex and a copy of K_{n-1}.

Theorem 3.1.4 is a consequence of the following lemma which explains a complementarity between the Laplacian spectra of a graph G and its complement.

Lemma 3.1.5. *Let G be a graph and let $0, \lambda_2, \dots, \lambda_n$ be the eigenvalues of $L(G)$, listed in increasing order. Then the eigenvalues of $L(\overline{G})$ are $0, n-\lambda_n, n-\lambda_{n-1}, \dots, n-\lambda_2$.*

Proof. That 0 is an eigenvalue of $L(\overline{G})$ is clear. Note that $L(G) + L(\overline{G}) = nI - J$, which is the Laplacian matrix of K_n. Suppose that $i \geqslant 2$ and let v be an eigenvector of G corresponding to λ_i. We may assume that $v \perp \mathbf{1}$ (where $\mathbf{1}$ denotes the all-1 vector). Thus the sum of the entries of v is zero. Then

$$L(\overline{G})v = (nI - J - L(G))v = nv - \lambda_i v = (n - \lambda_i)v.$$

Thus $n - \lambda_i$ is an eigenvalue of $L(\overline{G})$ whose eigenspace is the same as the $L(G)$-eigenspace of λ_i. □

From Lemma 3.1.5 is is immediate that n is an eigenvalue of $L(G)$ if and only if 0 occurs at least twice as an eigenvalue of $L(\overline{G})$, i.e. of and only if $\overline{G}$ is disconnected. The multiplicity of n as an eigenvalue of $L(G)$ is one less than the multiplicity of 0 as an eigenvalue of $L(\overline{G})$, i.e. one less than the number of connected components of $\overline{G}$.

Exercise: If n is an eigenvalue of $L(G)$ for some graph G, prove that 0 occurs only once as an eigenvalue of $L(G)$.