1.2 The Axioms of a Ring

Definition 1.2.1 *A* ring *is a non-empty set R equipped with two binary operations called addition $(+)$ and multiplication $(\times)$, satisfying the following properties :*

A1 Addition is associative.
$(r+s)+t = r+(s+t)$ for all $r, s, t \in R$.

A2 Addition is commutative. $r+s = s+r$ for all $r, s \in R$.

A3 R contains an identity element for addition, denoted by 0_R and called the *zero element* of R.
$r + 0_R = 0_R + r = r$ for all $r \in R$.

A4 Every element of R has an inverse with respect to addition. (The additive inverse of r is often denoted by $-r$).
For all $r \in R$, $\exists -r \in R$ for which $r + (-r) = 0_R$.

- Note : Axioms A1 to A4 could be summarized by saying that R is an abelian group under addition.

M1 Multiplication is associative.
$(r \times s) \times t = r \times (s \times t)$ for all $r, s, t \in R$.

D1 $r \times (s+t) = (r \times s) + (r \times t)$ for all $r, s, t \in R$.

D2 $(r+s) \times t = (r \times t) + (s \times t)$ for all $r, s, t \in R$.
-Distributive laws for multiplication over addition.

Remarks

1. A ring is called *commutative* if its multiplication is commutative.
2. A ring R is called *unital* or referred to as a *ring with identity* if it contains an identity element for multiplication. In this case we will denote the multiplicative identity by 1_R or just 1. An example of a ring without identity is $2\mathbb{Z}$.
3. The term "ring" was introduced by Hilbert in the late 19th century, when he referred to a "Zahlring" or "number ring".

We mention the following consequence of the ring axioms.

Lemma 1.2.2 *Let R be a ring. Then for all elements r of R we have*

$$0_R \times r = 0_R \text{ and } r \times 0_R = 0_R.$$

i.e. multiplying any element of R by the zero element results in the zero element as the product.

Proof : Let $r \in R$. We have

$$\begin{aligned} (0_R \times r) + (0_R \times r) &= (0_R + 0_R) \times r \\ &= 0_R \times r. \end{aligned}$$

Adding the additive inverse of the element $0_R \times r$ to both sides of this equation gives

$$0_R \times r = 0_R.$$

A similar argument shows that $r \times 0_R = 0_R$. $\square$