1.4 Integral Domains and Zero–Divisors

We saw in Lemma 1.2.2 that whenever an element of a ring is multiplied by zero, the result is zero. When working in the set of real numbers we often use the converse of this – a product $ab$ can be zero in $\mathbb{R}$ only if at least one of $a$ and $b$ is equal to zero (this is often used for example in solving polynomial equations by factoring).

**Question**: Is it true in every ring that the product of two elements can be zero only if at least one of the elements is zero?

**Example 1.4.1**  
1. In $M_2(\mathbb{Q})$

$$
\begin{pmatrix}
1 & -1 \\
-2 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.
$$

i.e. the product of two non-zero matrices in $M_2(\mathbb{Q})$ can be the zero matrix.

2. In $\mathbb{Z}/6\mathbb{Z}$, $2 \times 3 = 0$

So the answer to the question is no. However, people do study the class of rings in which the property described in the question holds.

**Definition 1.4.2** Let $R$ be a ring with zero element $0_R$. An element $a$ of $R$ is called a (left) zero–divisor in $R$ if $a \neq 0_R$ and there exists an element $b \neq 0_R$ of $R$ for which $ab = 0_R$. (In this case $b$ is a right zero–divisor).

**Note**: If $R$ is commutative then $ab = ba$ and we just talk about zero–divisors (not left and right zero–divisors).

**Definition 1.4.3** A commutative ring with identity that contains no zero-divisors is called an integral domain (or just a domain).

In an integral domain, the product of two elements can be zero only if one of the elements is zero.

**Examples**

1. $\mathbb{Z}$ is an integral domain. Somehow it is the “primary” example - it is from the ring of integers that the term “integral domain” is derived.

2. Every field is an integral domain. For let $F$ be a field and suppose that $a$, $b$ are elements of $F$ for which $ab = 0_F$. Assume $a \neq 0$. Then $a$ has a multiplicative inverse in $F$ and

$$
ab = 0_F \\
\Rightarrow a^{-1}(ab) = a^{-1}0_F \\
\Rightarrow (a^{-1}a)b = 0_F \text{ by Lemma 1.2.2} \\
\Rightarrow 1_F b = 0_F \\
\Rightarrow b = 0_F.
$$

**Remark**: It follows from the above argument that no unit can be a (left or right) zero-divisor in any ring.

3. An example of a commutative ring with identity that is not an integral domain is $\mathbb{Z}/6\mathbb{Z}$ (or $\mathbb{Z}/n\mathbb{Z}$ for any composite natural number $n$).