Aircraft Wing Vibrations During Landing

Jack Hampson
Jack McCarthy
Jack Roche
Shishir Suresh
Szymon Urbas
MENTOR: Giuseppe Zurlo

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Introduction

- Aeroplane hits the runway and landing gear causes shock
- Model wing as thin elastic beam
- Assume stationary
- At t = 0, gear acts

$$M \propto k \implies m = bk \ (b > 0)$$

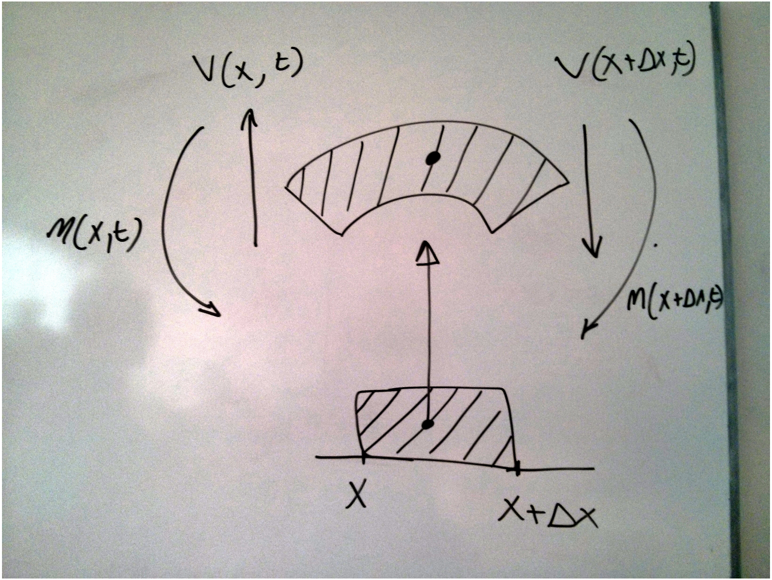


Figure: An elemental section of the 1 dimensional beam.

Rotational Equilibrium

In equilibrium, it is known that

$$V(x-\frac{\Delta x}{2},t)(-\frac{\Delta x}{2})-V(x+\frac{\Delta x}{2},t)(\frac{\Delta x}{2})+M(x-\frac{\Delta x}{2},t)-M(x+\frac{\Delta x}{2},t) =$$

$$(V(x,t)-\frac{\partial V}{\partial x}\frac{\Delta x}{2})(-\frac{\Delta x}{2})-(V(x,t)+\frac{\partial V}{\partial x}\frac{\Delta x}{2})(\frac{\Delta x}{2})+M(x,t)-\frac{\partial M}{\partial x}(\frac{\Delta x}{2}$$

$$V(x,t)\Delta x - \frac{\partial M}{\partial x}\Delta x = 0$$

$$V(x,t) = \frac{\partial M}{\partial x} \qquad (1)$$

Derivation of curvature and bending moment relationship

The curvature for a vector valued function is defined as,

$$\kappa := \left| \frac{d\boldsymbol{T}}{ds} \right|.$$

But for a function embedded in a plane with graph $y = f(x)$,

$$\kappa = \frac{|y''|}{(1 + y')^{3/2}}$$

It is assumed that slopes are small compared with unity, thus the curvature can be approximated as,

$$\kappa \approx \frac{d^2y}{dx^2} \tag{2}$$

As assumed

$$M = b\kappa = b\frac{\partial^2 u}{\partial x^2} \tag{3}$$

Kinematic equations

$$\boldsymbol{F}_{net} = m\boldsymbol{a}$$

$$
\begin{aligned}
V(x - \frac{\Delta x}{2}, t) - V(x + \frac{\Delta x}{2}, t) &= \rho \frac{\partial^2 u}{\partial t^2} \Delta x \\
V(x,t) - \frac{\partial V}{\partial x}\frac{\Delta x}{2} - V(x,t) + \frac{\partial V}{\partial x}\frac{\Delta x}{2} &= \rho \frac{\partial^2 u}{\partial t^2} \Delta x \\
-\frac{\partial V}{\partial x} &= \rho \frac{\partial^2 u}{\partial x^2 t} \\
\text{where} \quad V &= \frac{\partial M}{\partial x} \\
-\frac{\partial^2 M}{\partial x^2} &= \rho \frac{\partial^2 u}{\partial t^2} \\
M &= b\kappa \\
\implies -b\frac{\partial^2 \kappa}{\partial x^2} &= \rho \frac{\partial^2 u}{\partial t^2}
\end{aligned}
$$

Using the relationship in eq(2), it can be concluded that

$$\frac{\partial^4 u}{\partial x^4} = \left(-\frac{\rho}{b}\right) \frac{\partial^2 u}{\partial t^2} \tag{4}$$

Which is generally known as **Euler-Bernoulli beam equation**.

It is now assumed that there is a gravitational force $\rho \Delta x g$ acting on the infinitesimal element as considered in figure(2), when the forces are equated,

$$\begin{aligned} V(x - \frac{\Delta x}{2}, t) - V(x + \frac{\Delta x}{2}, t) - \rho \Delta x g &= \rho \Delta x \frac{\partial^2 u}{\partial t^2} \\ \Longrightarrow -\frac{\partial V}{\partial x} \Delta x &= \rho \Delta x \left(g + \frac{\partial^2 u}{\partial t^2} \right) \\ \frac{\partial V}{\partial x} &= \rho \Delta x \left(g + \frac{\partial^2 u}{\partial t^2} \right). \end{aligned}$$

Using eqs(1,2), the resulting equation is,

$$-b \frac{\partial^4 u}{\partial t^4} = \rho \left(g + \frac{\partial^2 u}{\partial t^2} \right)$$

Static solution

The **time-independent** static solution to the **Euler-Bernoulli beam equation** and the equation now becomes a $4th$ order ODE given by,

$$\frac{d^4u}{dx^4} = -\frac{\rho g}{b}$$

and the boundary conditions are given by,

$$\begin{aligned} u(0) &= 0 \\ u'(0) &= 0 \\ u''(L) &= 0 \\ u'''(L) &= 0. \end{aligned}$$

The solution to this IVP is given by,

$$u(x) = \frac{3g}{EL^3}\left(-\frac{1}{24}x^4 + \frac{L}{6}x^3 - \frac{L^2}{4}x^2\right)$$

Plot

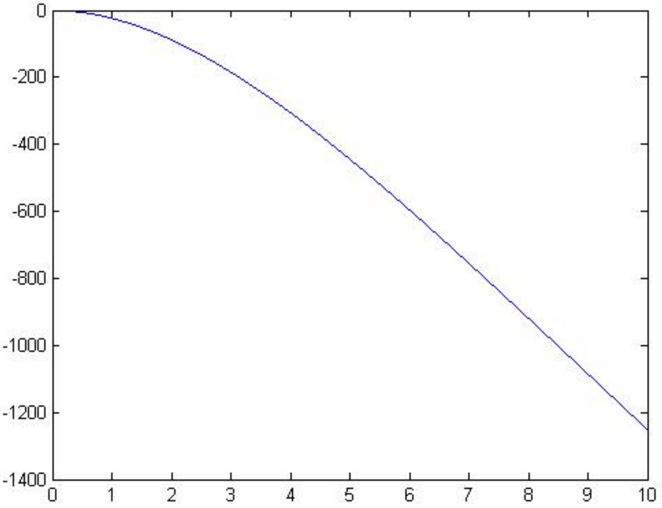


Figure: Non scaled plot of solution

Full Solution

We have

$$-b\frac{\partial^4 u}{\partial t^4} = \rho\left(g + \frac{\partial^2 u}{\partial t^2}\right)$$

$$\begin{aligned} u(0,t) &= 0 \\ u_x(0,t) &= 0 \\ u_{xx}(L,t) &= 0 \\ u_{xxx}(L,t) &= 0 \end{aligned}$$

$$u(x,0) = 0$$

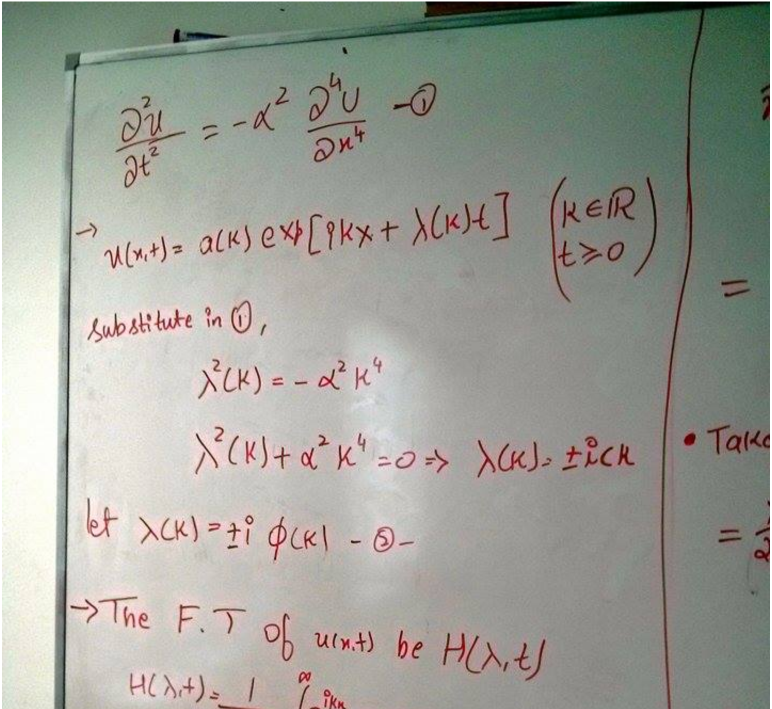
$$u_t(x,0) = 0$$

Using the separation of variables technique it we assume that the solution is of the form

$$u(x,t) = X(x)T(t) \tag{5}$$

We find that

$$u(x,t) = \sum_{n=1}^{\infty} A_n[[cosh(\beta_n x) - cos(\beta_n x)] - \frac{\omega}{\gamma}[sinh(\beta_n x) - sin(beta_n x)]]sin(\alpha$$



∂²u/∂t² = −α² ∂⁴u/∂x⁴ —(1)
→ u(x,t) = a(k) exp[ikx + λ(k)t]
(k ∈ ℝ, t ≥ 0)
Substitute in (1),
λ²(k) = −α²k⁴
λ²(k) + α²k⁴ = 0 ⇒ λ(k) = ±iαk
let λ(k) = ±i φ(k) -(2)-
→ The F.T of u(x,t) be H(λ,t)
• Tak

This is a very classical approach to solve the 1-d beam equation, however it does not reveal the boundary conditions and the physical nature of the problem.

QUESTIONS?