Spread and Uniform Domination

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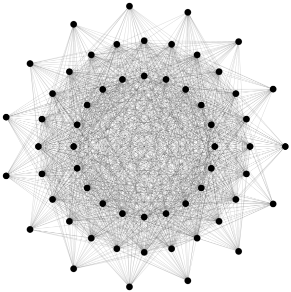
(joint with Tim Burness)

University of Bristol

Groups in Galway

NUI Galway

11th May 2019



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Question How are the generating pairs distributed across the group?

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- (uniform) spread
- (uniform) domination number

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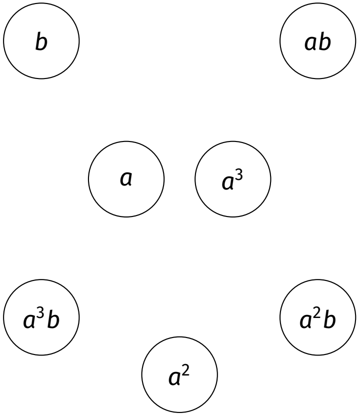

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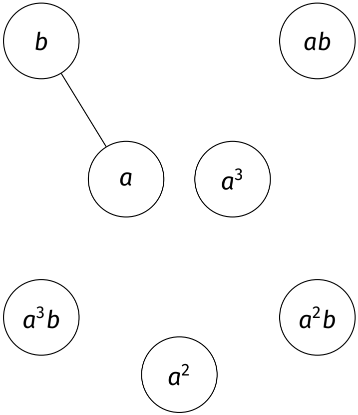

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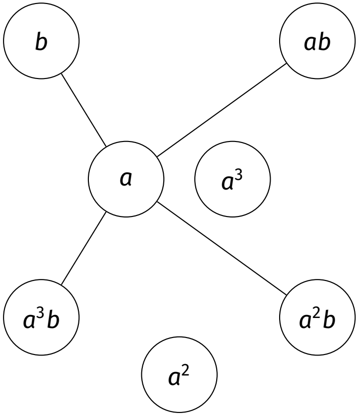

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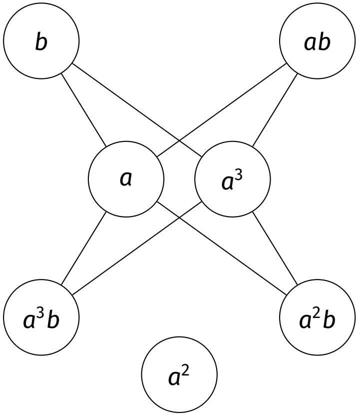

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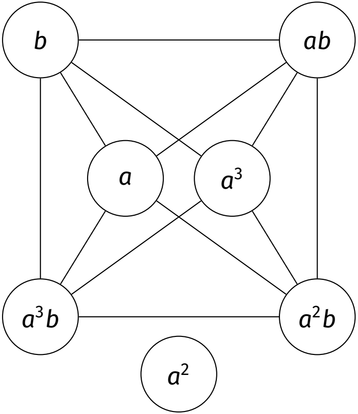

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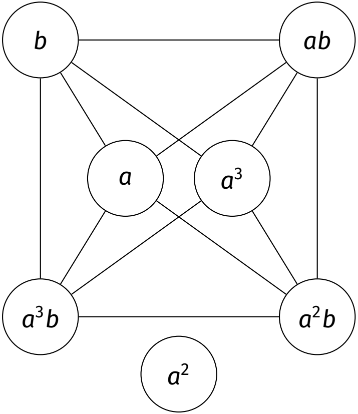
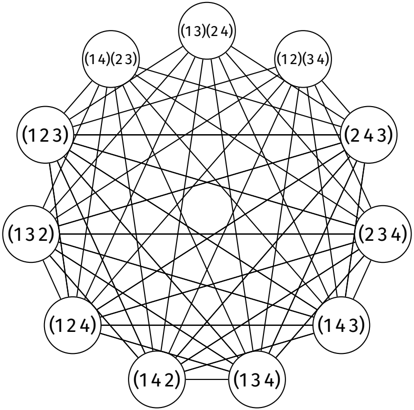


Alternating group A_4



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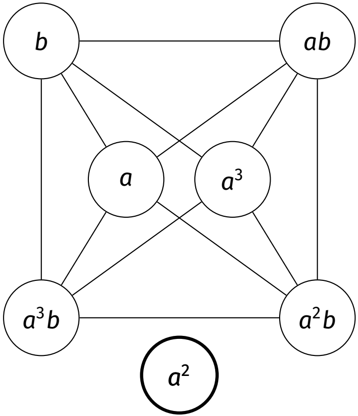

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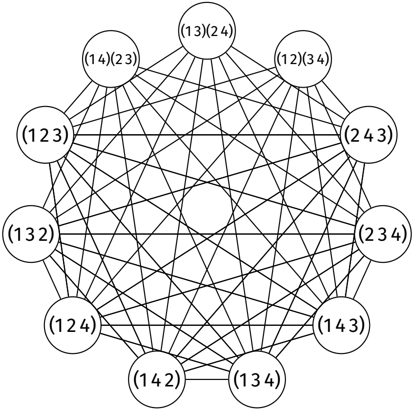

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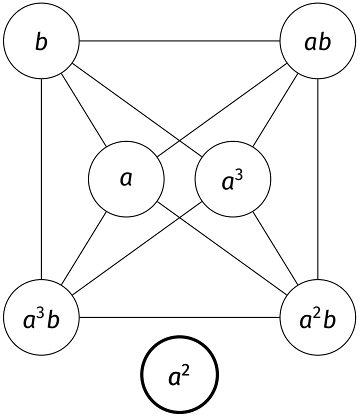

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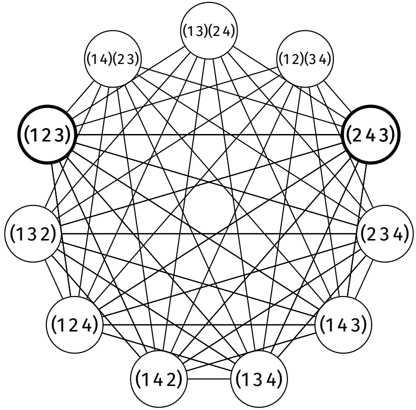

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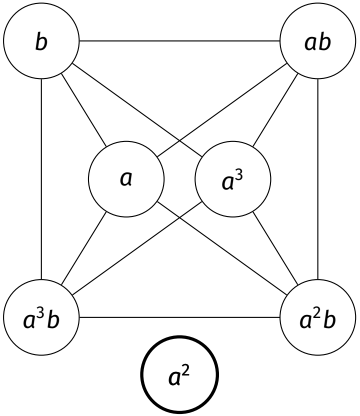

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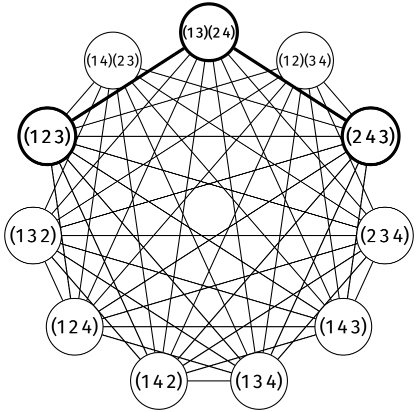

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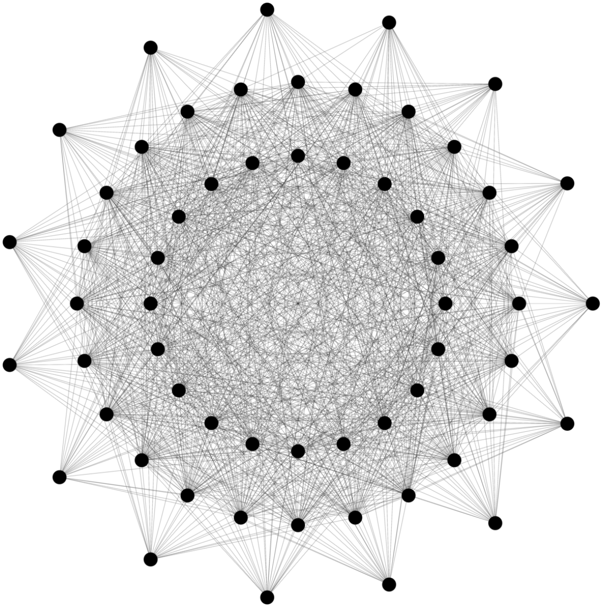

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The **spread** of G, written $\boldsymbol{s(G)}$, is the greatest integer k such that for all $x_1, \ldots, x_k \in G \setminus 1$, there exists $y \in G$ such that $\langle x_1, y \rangle = \cdots = \langle x_k, y \rangle = G$.

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The **uniform spread** of G, written $u(G)$, is the greatest integer k such that there exists a conjugacy class C such that for all $x_1, \dots, x_k \in G \setminus 1$, there exists $y \in C$ such that $\langle x_1, y \rangle = \cdots = \langle x_k, y \rangle = G$.

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  - alternating groups of degree divisible by a fixed prime
  - symplectic groups over a field of fixed even size
  - odd-dimensional orthogonal groups over a field of fixed size

Spotlight on $\mathrm{PSL}_2(q)$

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Let $G = \mathrm{PSL}_2(q)$ with $q \geqslant 11$.

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Remark (Burness & H, 2018)

Let q be a **prime** satisfying $q \equiv 3 \pmod 4$. Then $s(G) - u(G) = \frac{1}{2}(q + 1)$.

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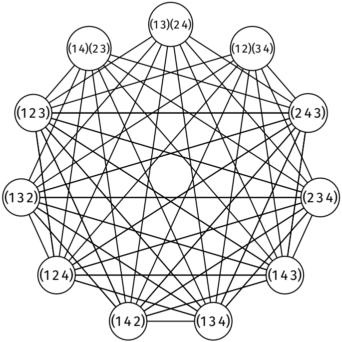

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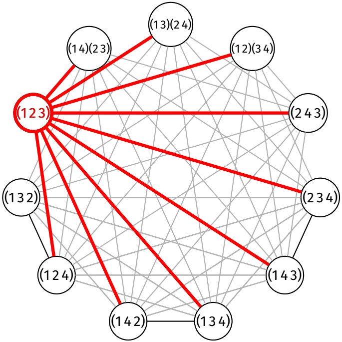

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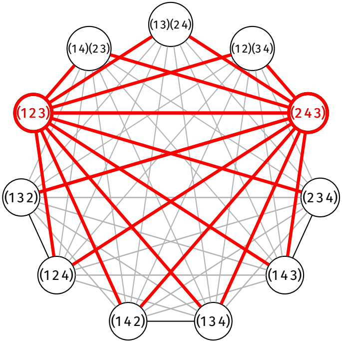

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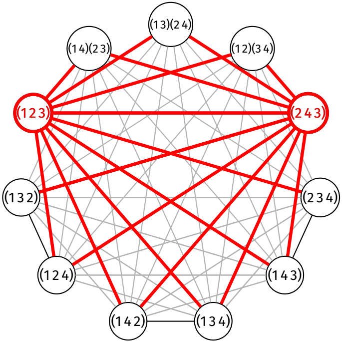

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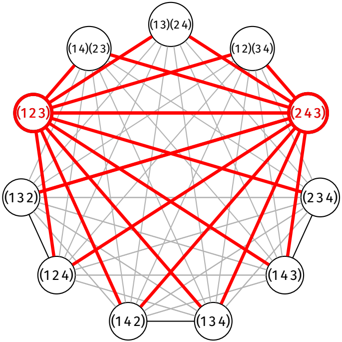

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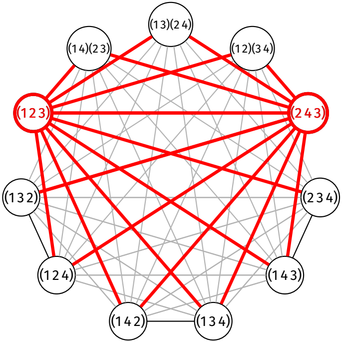

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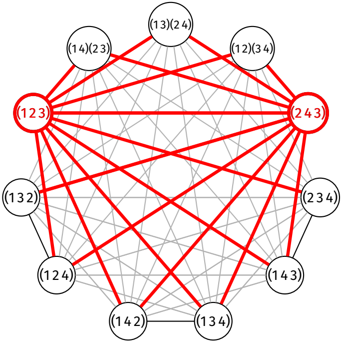

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$$2 \leqslant \gamma_t(G) \leqslant \gamma_u(G) \leqslant |C|$$

for a particular (typically quite large) conjugacy class C of G.

Domination

The **total domination number** of G, written $\gamma_t(G)$, is the smallest size of a set $T \subseteq G$ such that for all $x \in G \setminus 1$, there exists $y \in T$ such that $\langle x, y \rangle = G$.

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Question When is $\gamma_u(G) = 2$?

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Let G be an almost simple group with a primitive nonstandard action on Ω. Then $b(G, \Omega) \leqslant 7$ with equality iff $G = \mathrm{M}_{24}$ and $|\Omega| = 24$.

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Observation If $H \leqslant G$ is core-free, then G acts faithfully on G/H and $\{Hg_1, \ldots, Hg_c\}$ is a base iff $\bigcap_{i=1}^{c} H^{g_i} = 1$.

Method: Base Sizes

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In particular, $\gamma_u(G) \leqslant b(G, G/H)$.

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$$P(G, s, c) \geqslant 1 - \sum_{i=1}^{k} |x_i^G| \left(\sum_{H \in \mathcal{M}(G,s)} \frac{|x_i^G \cap H|}{|x_i^G|} \right)^c$$

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If $\mathcal{M}(G, s) = \{H\}$, then the Probabilistic Lemma is the probabilistic approach introduced by Liebeck and Shalev for base sizes.

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Therefore $\gamma_u(G) \leqslant 2r + 26$.

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- $\gamma_u(G) \leqslant 80 \log_2 n$

  [n is even $\implies \log_2 n \leqslant \gamma_t(A_n) \leqslant \gamma_u(A_n) \leqslant 2 \log_2 n$]
- $\gamma_u(G) = 2 \iff n \geqslant 13$ is prime
- $\gamma_u(G) = 2 \implies P_2(G) > 1 - n^{-1}$ (unless $n = 13$)

Spread and Uniform Domination

Scott Harper

(joint with Tim Burness)

University of Bristol

Groups in Galway

NUI Galway

11th May 2019