Some decomposition matrices of finite classical groups

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- $\mathbb{K}$ a finite extension of $\mathbb{Q}_\ell$
- $\mathcal{O} \subset \mathbb{K}$ ring of integers
- $\Bbbk = \mathbb{K}/\mathcal{O}$, $\mathrm{char}\ \Bbbk = \ell$

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characteristic 0		integral lattice		characteristic ℓ
$\rho \in \mathrm{Irr}\, \mathbb{K}G\text{-mod}$	$\rightsquigarrow$	$\Lambda_\rho \in \mathcal{O}G\text{-mod}$	$\rightsquigarrow$	$\Lambda_\rho \otimes \Bbbk \in \Bbbk G\text{-mod}$

Modular representation theory: baby examples

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1. Non-isomorphic representations in characteristic 0 may become isomorphic in characteristic ℓ: take $G = S_2 \cong \langle s \mid s^2 = 1 \rangle$ and $\ell = 2$.
   $\rho_{(2)} = \mathrm{triv}_{\mathbb{K}} \cong \mathbb{K}$ on which s acts by 1 reduces to the trivial representation mod 2,
   $\rho_{(1,1)} = \mathrm{sign}_{\mathbb{K}} \cong \mathbb{K}$ on which s acts by -1 also reduces to the trivial representation mod 2, since $-1 = +1 \mod 2$.

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2. ρ may not stay irreducible mod ℓ: take $G = S_3$ and $\ell = 3$.
   $\rho_{(2,1)}$ the reflection rep of $\mathbb{K}S_3$. Integral version of $\rho_{(2,1)}$: $\{(x_1, x_2, x_3) \in \mathcal{O}^3 \mid x_1 + x_2 + x_3 = 0\}$. Mod 3 it has the trivial representation $\mathrm{triv}_{\Bbbk}$ as a submodule and the sign representation $\mathrm{sign}_{\Bbbk}$ as a quotient.

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Abusing notation, we will also use the notation ρ in this talk for the ℓ-reduction of the integral form of ρ, and talk about what simple composition factors ρ has a $\Bbbk G$-representation.

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Decomposition numbers $d_{\rho,\phi}$:

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The decomposition numbers may be assembled into a matrix:

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It's very difficult to find decomposition numbers in general despite powerful methods from Lie theory. E.g. the decomposition matrix of S_n for an arbitrary prime $\ell \leq n$ is not known.

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There is a distinguished subset of $\mathrm{Irr}\,\mathbb{K}G(q)$ called *unipotent representations*, defined using geometry related to the flag variety of $\mathbf{G}$ (Deligne-Lusztig varieties).

$|\mathrm{Irr}\,\mathbb{K}G(q)| \to \infty$ as $q \to \infty$ but $|\{\text{unip reps of } \mathbb{K}G(q)\}| < \infty$, indep of q.

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The unipotent decomposition matrix of $G(q)$ in characteristic ℓ is the submatrix of the decomposition matrix of $G(q)$ given by $D = (d_{\rho,\phi})$ s.t. ρ is unipotent, $\phi \in \mathrm{Irr}\,\Bbbk G(q)$ s.t. $d_{\rho,\phi} \neq 0$ for some unip ρ.

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Expectation: the decomposition matrix of $G(q)$ can be recovered from D.

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Revised problem

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Example: $G(q) = \mathrm{GL}_3(q)$, {unip reps of $\mathrm{GL}_3(q)$} $\overset{1:1}{\leftrightarrow}$ {partitions of 3}.
Take $\ell \mid \Phi_3(q) = q^2 + q + 1$. Then:

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Properties of the unipotent decomposition matrix D:

- the matrix D is square and invertible (Geck-Hiss),
- expected by experts but open: for ℓ large enough, D does not depend on q but only on the order of q mod ℓ,
- D is lower unitriangular (conj: Geck; proof: Brunat-Dudas-Taylor) $\rightsquigarrow$ labels for $\{\phi \in \mathrm{Irr}\, \Bbbk G(q) \mid d_{\rho,\phi} \neq 0 \text{ some unip } \rho\}$.

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$$|G(q)| = q^{|R|} \prod_{i=1}^{m} (q^{d_i} - 1)$$

where d_i are the degrees of the generators of $S(\mathfrak{h})^W$.

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The more times $\Phi_d(q)$ divides $|G(q)|$, the more difficult to determine D.

Methods for computing decomposition matrices of FGLT

1. Algebraic - Harish-Chandra induction/restriction to produce new characters from old, connection to Hecke algebras
2. Geometric - Deligne-Lusztig varieties produce projective characters via cohomology, give info about cuspidals (reps that can't obtained by HC induction)
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$$\begin{array}{c} \\ 2. \\ .2 \\ B_2 \\ 1^2. \\ .1^2 \end{array} \begin{array}{c} \begin{array}{ccccc} 2. & .2 & B_2 & 1^2. & .1^2 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 1 & 1 \end{pmatrix} \end{array}$$

The case $\ell \mid \Phi_{2n}(q)$ for $\mathrm{Sp}_{4n}(q)$ and $\mathrm{SO}_{4n+1}(q)$

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- $2n + 2 \leq d \leq 4n$, d even $\implies \Phi_d(q) \mid |G_{2n}(q)|$ once $\implies D$ known;

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Cases $G_4(q)$ and $\ell \mid \Phi_4(q)$, $G_6(q)$ and $\ell \mid \Phi_6(q)$: dec matrix found by Dudas-Malle.

Theorem (Dudas-N., '20)

The unipotent decomposition matrix of $\mathrm{Sp}_{4n}(q)$ or $\mathrm{SO}_{4n+1}(q)$ is known when $\ell \mid \Phi_{2n}(q)$.

Example: dec matrix of $\mathrm{Sp}_{12}(q)$ or $\mathrm{SO}_{13}(q)$ for $\ell \mid \Phi_6(q)$

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It is also given by our theorem.
For $G_n(q) = \mathrm{Sp}_{2n}(q)$ or $\mathrm{SO}_{2n+1}(q)$, the unipotent characters of $G_n(q)$ are parametrized in "series" by:

- bipartitions of n (principal series),
- bipartitions of $n-2$ (B_2 series),
- bipartitions of $n-6$ (B_6 series),
- ...
- bipartitions of $n-(t^2+t)$ (B_{t^2+t} series)

so long as $n-(t^2+t) \geq 0$.

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so long as $n-(t^2+t) \geq 0$. For $G_6(q)$, three "series" of unipotent characters: principal series (B_0) $\lambda^1.\lambda^2$ with $|\lambda^1|+|\lambda^2| = 6$, B_2 series $\mu^1.\mu^2$ with $|\mu^1|+|\mu^2| = 6-2 = 4$, B_6 series $\nu^1.\nu^2$ with $|\nu^1|+|\nu^2| = 6-6 = 0$.

6.	1																										
.6		1																									
51.	1		1																								
2.4	1	1		1																							
$B_2 : 2^2.$					1																						
$41^2.$			1			1																					
.51		1					1																				
21.3	1		1	1				1																			
1.41		1		1			1		1																		
$B_2 : 21.1$					1					1																	
B_6											1																
$21^2.2$			1			1		1				1															
$1^2.31$				1				1	1				1														
$B_2 : 1^2.2$										1				1													
$31^3.$						1									1												
$.41^2$							1		1							1											
$B_2 : 2.1^2$										1							1										
$21^3.1$						1						1			1			1									
$1^3.21$								1				1	1						1								
$B_2 : 1.21$					1					1				1			1			1							
$B_2 : .2^2$					1															1	1						
$1^4.1^2$												1						1	1			1					
$21^4.$															1			1					1				
$.31^3$									1				1			1								1			
$1^6.$											2							1				1	1		1		
$.21^4$									1										1					1		1	
$.1^6$																			1		2	1				1	1

THANK YOU!