AN ALGORITHM FOR THE UNIT GROUP OF THE BURNSIDE RING OF A FINITE GROUP.

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ABSTRACT. In this note we present an algorithm for the construction of the unit group of the Burnside ring $\Omega(G)$ of a finite group G from a list of representatives of the conjugacy classes of subgroups of G.

1. INTRODUCTION.

Let G be a finite group. The Burnside ring $\Omega(G)$ of G is the Grothendieck ring of the isomorphism classes of the finite right G-sets with respect to disjoint union and direct product. It has a $\mathbb{Z}$-basis consisting of the isomorphism classes of the transitive G-sets G/H, where H runs through a system of representatives of the conjugacy classes of subgroups of G.

The ghost ring of G is the set $\tilde{\Omega}(G)$ of functions f from the set of subgroups of G into $\mathbb{Z}$ which are constant on conjugacy classes of subgroups of G. For any finite G-set X, the function ϕ_X which maps a subgroup H of G to the number of its fixed points on X, i.e., $\phi_X(H) = \#\{x \in X : x.h = x \text{ for all } h \in H\}$, belongs to $\tilde{\Omega}(G)$. By a theorem of Burnside, the map $\phi : [X] \to \phi_X$ is an injective homomorphism of rings from $\Omega(G)$ to $\tilde{\Omega}(G)$. We identify $\Omega(G)$ with its image under ϕ in $\tilde{\Omega}(G)$, i.e., for $x \in \Omega(G)$, we write $x(H) = \phi(x)(H) = \phi_H(x)$.

The ghost ring has a natural $\mathbb{Z}$-basis consisting of the characteristic functions of the conjugacy classes of subgroups of G. The table of marks of G is defined as the square matrix $M(G)$ which records the coefficients when the transitive G-sets G/H are expressed as linear combinations of the characteristic functions. If G has r conjugacy classes of subgroups $M(G)$ is an $r \times r$ matrix over $\mathbb{Z}$ which is invertible over $\mathbb{Q}$.

Let $H_1, \ldots, H_r$ be representatives of the conjugacy classes of subgroups of G. Then we can further identify the ghost ring $\tilde{\Omega}(G)$ with $\mathbb{Z}^r$, where, for $x \in \tilde{\Omega}(G)$, we set $x_i = x(H_i)$, $i = 1, \ldots, r$. For $x \in \mathbb{Z}^r$, the product $xM(G)^{-1}$ yields the multiplicities of the transitive G-sets G/H_i in x. An element $x \in \tilde{\Omega}(G)$ thus lies in $\Omega(G)$ if and only if $xM(G)^{-1}$ consists of integers only.

The units of the ghost ring are $\{\pm 1\}^r \subseteq \mathbb{Z}^r$. We want to determine those ± 1-vectors which are contained in the image of $\Omega(G)$ in $\mathbb{Z}^r$. Of

course every such vector can be tested with the table of marks. But this task grows exponentially with the number r of conjugacy classes.

The following result of Yoshida [4] gives a necessary and sufficient condition, which will allow us to explicitly calculate a basis of the unit group $\Omega^*(G)$ of $\Omega(G)$.

1.1. Theorem. *Let u be a unit in $\tilde{\Omega}(G)$. Then $u \in \Omega(G)$ if and only if, for every subgroup $H \leq G$, the function $\mu^H : N_G(H) \to \mathbb{C}$ defined by $\mu^H(n) = u(H \langle n \rangle)/u(H)$ is a linear character of $N_G(H)$.*

Here $u(H \langle n \rangle)$ is the value of u at the preimage in $N_G(H)$ of the cyclic subgroup of $N_G(H)/H$ generated by the coset Hn. The Theorem follows from a more general characterization of elements of the ghost ring which lie in the Burnside ring by certain congruences.

2. The algorithm.

Let E be an elementary abelian 2-group of order 2^m, generated by $e_1, \ldots, e_m$. Every linear character λ of E is determined by its values on the e_i, which in turn can be chosen, independently, to be $+1$ or -1.

Given a subgroup $H \leq G$, to say that μ^H is a linear character of $N = N_G(H)$ amounts to the following. First, let $R \leq N$ be the minimal subgroup such that $H \leq R$ and N/R is an elementary abelian 2-group. Since μ^H has only values ± 1 it must have R in its kernel and can be regarded as a character of the elementary abelian 2-group $E := N/R$. Let $e_1, \ldots, e_m$ be a basis of E.

Let $n \in N$ and consider the coset $Hn \in N/H$. The element $Rn \in E$ can be expressed in a unique way as linear combination $Rn = e_1^{\alpha_1} \cdots e_m^{\alpha_m}$, with $\alpha_k \in \{0, 1\}$, $k = 1, \ldots, m$.

Let λ be a linear character of E. Then λ is determined by the values $\lambda(e_k)$, $k = 1, \ldots, m$ and $\lambda(Rn) = \lambda(e_1)^{\alpha_1} \cdots \lambda(e_m)^{\alpha_m}$.

Now μ^H is a linear character if and only if $\mu^H = \lambda$ for some choice of the values $\lambda(e_k)$, $k \in 1, \ldots, m$, i.e., $\mu^H(n) = \lambda(Rn) = \lambda(e_1)^{\alpha_1} \cdots \lambda(e_m)^{\alpha_m}$. Thus u must satisfy

$$(1) \qquad u(H \langle n \rangle)/u(H) = \lambda(e_1)^{\alpha_1(n)} \cdots \lambda(e_m)^{\alpha_m(n)}.$$

Let $p, q \in \{1, \ldots, r\}$ be such that H is a conjugate of H_p and $H \langle n \rangle$ is a conjugate of H_q. Then (1) can be written as a linear equation over $GF(2)$ in the unknowns $l_1, \ldots, l_m$ (such that $\lambda(e_k) = (-1)^{l_k}$, $k = 1, \ldots, m$), and $v_1, \ldots, v_r$ (such that $u(H_i) = (-1)^{v_i}$, $i = 1, \ldots, r$) as

$$(*) \qquad \alpha_1 l_1 + \cdots + \alpha_m l_m + v_p + v_q = 0.$$

For a given subgroup $H \leq G$, each coset $Hn \in N/H$ contributes one such equation; conjugate elements of N/H of course yield the same equation. Since n can be chosen such that $Rn = e_k$, the system contains equations of the form

$$l_k + v_p + v_q = 0,$$

which allow us to express the l_k in terms of the v_i, for all $k = 1, \ldots, m$. What remains, for each subgroup H, is a (possibly trivial) system of homogeneous equations in the v_i only, which we denote by $\mathcal{E}(H)$. Of course, conjugate subgroups give rise to the same system of equations. The following theorem is now immediate.

2.1. **Theorem.** $u \in \Omega(G)$ *if and only if, for each subgroup $H \leq G$, it satisfies the conditions $\mathcal{E}(H)$.*

The algorithm is based on Theorem 2.1. Given a list $H_1, H_2, \ldots H_r$ of representatives of subgroups of G, the following steps are taken for each $H = H_i$, $i = 1, \ldots, r$.

1. Let $N = N_G(H)$ and $Q = N/H$. Let q_j, $j = 1, \ldots, l$, be representatives of the conjugacy classes of Q and let $C_j = H \langle q_j \rangle$ be the subgroup of G corresponding to the cyclic subgroup of Q generated by q_j. Then C_j is a conjugate of some H_k and $u(C_j) = u(H_k)$ for all $u \in \mathbb{Z}^r$.
2. Let $H \leq R \leq N$ be such that $E := N/R$ is the largest elementary abelian 2-quotient of N/H. Inside G, this subgroup R can be found as closure of H, the derived subgroup N' and the squares g^2 of all generators g of N.
3. Regard E as an $GF(2)$-vector space and find a basis $e_1, \ldots, e_m$. (This requires a search through the elements of E until a large enough linearly independent set has been found.) Now every element $e \in E$ can be described as a unique linear combination $e = \alpha_1 e_1 + \cdots + \alpha_m e_m$ of the basis elements with $\alpha_i \in \{0, 1\}$. In particular, for every representative q_j, we get such a decomposition of the coset $Rq_j \in E$.
4. For each q_j write down its equation $(*)$. Then eliminate the unknowns l_k to yield $\mathcal{E}(H)$.

Finally, it remains to solve the system $\bigcup_{i=1}^{r} \mathcal{E}(H_i)$: its nullspace corresponds to the group of units $\Omega^*(G)$.

3. Examples.

Theorem 1.1 can be used to determine the units of the Burnside ring of an abelian group.

3.1. **Theorem.** *If G is a finite abelian group whose largest elementary abelian 2-quotient has order 2^n, then $|\Omega^*(G)| = 2^{2^n}$. In particular, if G is an elementary abelian 2-group of order 2^n then $|\Omega^*(G)| = 2^{2^n}$.*

Proof. Let $N_1, \ldots, N_{2^n - 1} \leq G$ be the (maximal) subgroups of index 2 in G and define $\lambda_i \in \mathbb{Z}^r$ for $i = 1, \ldots, 2^n - 1$ as

(2) $$\lambda_i(H) = \begin{cases} +1 & \text{if } H \leq N_i, \\ -1 & \text{otherwise.} \end{cases}$$

Furthermore set $\lambda_G := \prod_{i=1}^{2^n-1} \lambda_i$ if $n \geq 1$ and $\lambda_G := -1$ if $n = 0$. Then

$$(3) \qquad \lambda_G(H) = \prod_{i=1}^{2^n-1} \lambda_i(H) = \begin{cases} -1 & \text{if } H = G, \\ +1 & \text{if } H \in \{N_1, \ldots, N_{2^n-1}\}. \end{cases}$$

We claim that the 2^n units $\mathcal{B} = \{-\lambda_i : i = 1, \ldots, 2^n - 1\} \cup \{\lambda_G\}$ form a basis of $\Omega^*(G)$.

First, we show that $\lambda_i \in \Omega(G)$. Fix $H \leq G$ and denote by μ^H the function $N_G(H) \to \mathbb{C}$ as defined in Theorem 1.1 for $u = \lambda_i$. Now, if $H \not\leq N_i$ then $U \not\leq N_i$ for all U with $H \leq U \leq G$. Hence $\lambda_i(U)/\lambda_i(H) = 1$ for all such U, i.e, μ^H is the trivial character of G/H. And if $H \leq N_i$ then μ^H is the linear character of G/H with kernel N_i/H. In any case, μ^H is a linear character of G/H, and from Theorem 1.1 then follows that $\lambda_i \in \Omega(G)$. Together with $-1 \in \Omega(G)$ this yields $\mathcal{B} \subseteq \Omega(G)$.

Next, note that $\mathcal{B}$ is linearly independent. For each such function, restricted to $\{N_i : i = 1, \ldots, 2^n - 1\} \cup \{G\}$ has exactly one value equal to -1.

Finally, every unit $u \in \Omega^*(G)$ is a linear combination of the $-\lambda_i$, $i = 1, \ldots, 2^n - 1$, and λ_G. For the values of u at $\{N_i : i = 1, \ldots, 2^n - 1\} \cup \{G\}$ determine a unique linear combination v of $\mathcal{B}$ which coincides with u on $\{N_i : i = 1, \ldots, 2^n - 1\} \cup \{G\}$. Now it suffices to show that for every subgroup $H \leq G$ with $|G : H| > 2$ and for every unit w of $\Omega(G)$, the value $w(H)$ is already determined by the values $w(U)$ for subgroups U of G with $H < U$. To see this note that there must exist a subgroup U of G containing H such that U/H is either of odd prime order, or cyclic of order 4, or elementary abelian of order 4. From Theorem 1.1 we obtain a linear character μ^H on U/H with values ± 1. In the first case, this character is trivial which implies $w(H) = w(U)$. In the second case this character must be trivial on the subgroup V/H of U/H of order 2. This implies $w(H) = w(V)$. In the third case, observe that every linear character μ of U/H satisfies $\mu(U_1/H)\mu(U_2/H)\mu(U_3/H) = 1$, where $U_1/H, U_2/H, U_3/H$ are the subgroups of order 2 of U/H. This implies $w(H) = w(U_1)w(U_2)w(U_3)$.

$\square$

The argument which shows the linear independence of the set $\mathcal{B}$ is still valid in a general 2-group. Thus $\mathrm{rk}\,\Omega^*(G) \geq 2^n$ for any 2-group G with $|G/\Phi(G)| = 2^n$. It may however happen that $|N_G(H) : H| < 4$ and then the argument which shows that $\mathcal{B}$ spans the unit group breaks down. In fact, if G is the dihedral group of order 8 then $|G/\Phi(G)| = 4$ but $\mathrm{rk}\,\Omega^*(G) = 5$.

Let A be a finite abelian group of odd order, and let $i : A \to A$ be the automorphism of A which maps every element to its inverse, $i(a) = a^{-1}$, $a \in A$. Then let G be the semidirect product of A and $\langle i \rangle$. The conjugacy classes of subgroups of G are easy to describe in terms of the subgroups of A. For every subgroup N of A there are two

conjugacy classes of subgroups of G. One consists of N only, since N is normal in G, and the other consists of $|A : N|$ conjugates of $\langle N, i \rangle$, which is a self-normalizing subgroup of G.

Let $u \in \Omega^*(G)$. It follows from Theorem 1.1 and the fact that the normalizer of every $N \leq A$ is G, that u is constant on $\{N : N \leq A\}$. Moreover, it is easy to see that for every $N \leq A$, the function $u_N \in \tilde{\Omega}(G)$ defined by

$$u_N(H) = \begin{cases} -1 & \text{if } H =_G \langle N, i \rangle, \\ 1 & \text{otherwise,} \end{cases}$$

is a unit in $\Omega(G)$. Thus $\mathrm{rk}_{\mathsf{GF}(2)} \, \Omega^*(G) = r + 1$, where r is the number of subgroups of A.

An implementation of the algorithm from section 2 in the GAP system for computational discrete algebra [3] allows us to calculate $\Omega^*(G)$ for particular groups G, given a list of representatives of the conjugacy classes of subgroups of G. GAP contains programs to calculate such a list for small groups. A procedure for the construction of a list of representatives of classes of subgroups (as well as the complete table of marks) of almost simple groups G has been described in [2].

The following table shows some of the results obtained.

G	$\mathrm{rk}\,\Omega^*(G)$	G	$\mathrm{rk}\,\Omega^*(G)$	G	$\mathrm{rk}\,\Omega^*(G)$	G	$\mathrm{rk}\,\Omega^*(G)$
A_3	1	S_3	3	M_{11}	18	J_1	15
A_4	2	S_4	6	M_{12}	49	J_2	38
A_5	5	S_5	10	M_{22}	59		
A_6	12	S_6	23				
A_7	20	S_7	34				
A_8	44	S_8	67				
A_9	66	S_9	110				

4. A CONJECTURE.

Let $\Omega_2(G)$ be the ring of monomial representations of G which are induced from linear representations of subgroups which have values ± 1 only. Then $\Omega_2(G)$ is a subring of the ring of all monomial representations of G containing the Burnside ring $\Omega(G)$. It has a basis labeled by the conjugacy classes of pairs (H, λ), where λ is a linear character of H with $\lambda(h) = \pm 1$ for all $h \in H$, or equivalently labeled by the conjugacy classes of pairs (H, K) where $K \leq H$ is such that $|H : K| \leq 2$ (corresponding to the kernel of λ).

4.1. Conjecture. *Let G be a finite group. Then*

$$\mathrm{rk}\,\Omega^*(G) - 1 \leq \mathrm{rk}\,\Omega_2(G) - \mathrm{rk}\,\Omega(G).$$

Using a result of Dress, the conjecture would imply immediately that any group G of odd order is solvable. For, if $|G|$ is odd no subgroup of G

has a non-trivial linear character with values ± 1 or, equivalently, a subgroup of index 2. Hence $\mathrm{rk}\,\Omega_2(G) = \mathrm{rk}\,\Omega(G)$ and thus $\Omega^*(G) = \{\pm 1\}$. But if $\Omega(G)$ contains no non-trivial units, then it contains no non-trivial idempotents either (because a non-trivial idempotent e yields a non-trivial unit $2e - 1$). Solvability of G then follows by Dress' characterisation of solvable groups [1].

The formula clearly holds for 2-groups: if G is a 2-group then every non-trivial subgroup $H \leq G$ has a subgroup of index 2, whence $\Omega_2(G) - \mathrm{rk}\,\Omega(G) \geq \mathrm{rk}\,\Omega(G) - 1$. On the other hand, one always has $\mathrm{rk}\,\Omega^*(G) \leq \mathrm{rk}\,\Omega(G)$.

Of course most often a nontrivial subgroup H has many more than just one subgroup of index 2. In fact, for an elementary abelian group G of order 2^n one has

$$\mathrm{rk}\,\Omega_2(G) - \mathrm{rk}\,\Omega(G) = [n]_2 \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_2,$$

where $[k]_q = \frac{1-q^k}{1-q}$ and $[k]_q! = [1]_q [2]_q \cdots [k]_q$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}$. Thus, in this case, $\mathrm{rk}\,\Omega_2(G) - \mathrm{rk}\,\Omega(G)$ is a large multiple of $\mathrm{rk}\,\Omega^*(G) - 1 = [n]_2$. It follows from Theorem 3.1 that the conjecture is true for abelian groups. In fact, if G has odd order, this is clear; and if G has even order, let G/N be the largest elementary abelian 2-factor group and assume it has order 2^n. Then, using Theorem 3.1,

$$\mathrm{rk}\,\Omega^*(G) - 1 = |G/N| - 1 = [n]_2$$

$$\leq [n]_2 \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_2 = \mathrm{rk}\,\Omega_2(G/N) - \mathrm{rk}\,\Omega(G/N)$$

$$\leq \mathrm{rk}\,\Omega_2(G) - \mathrm{rk}\,\Omega(G),$$

where the last inequality follows from the fact that to each pair of subgroups $K/N \leq H/N$ of G/N such that K/N has index 2 in H/N corresponds at least one such pair (namely $K \leq H$) of subgroups of G.

The Feit-Thompson Theorem implies the conjecture for groups of odd order. Clearly there are no subgroups of index 2 in a group of odd order. Moreover, such a group admits only the trivial units in its Burnside ring, see Lemma 6.7 [4].

If G is the semidirect product of an abelian group A of odd order and the inversion i, we have seen in Section 3 that $\mathrm{rk}_{\mathrm{GF}(2)}\,\Omega^*(G) = r + 1$, where r is the number of subgroups of A. Now each subgroup N of A occurs as a subgroup of index 2 in $\langle H, i \rangle$. It follows that $\mathrm{rk}\,\Omega_2(G) - \mathrm{rk}\,\Omega(G) = r$. So this class of groups provides infinitely many examples where the inequality in the conjecture becomes an equality. The only other known such example is the alternating group A_5.

In a slightly more general situation, let us suppose G has order $2m$ for an odd $m \in \mathbb{N}$. Then, using Feit-Thompson, G is solvable. Moreover, $\mathrm{rk}_{GF(2)} \, \Omega^*(G)$ equals the number of representatives H of conjugacy classes of subgroups of G which have no normal subgroup of index p for an odd prime p, see again Lemma 6.7 [4]. On the other hand $\mathrm{rk}\,\Omega_2(G) - \mathrm{rk}\,\Omega(G) = r$ equals the number of representatives H of conjugacy classes of subgroups of G which have a normal subgroup of index 2. Since, in a solvable group, every nontrivial subgroup has a normal subgroup of prime index, each representative which has no normal subgroup of index p for an odd prime p must have one of index 2. This shows the conjecture in that case.

And if G is a solvable group, it is still true that $\mathrm{rk}\,\Omega^*(G)$ is less than or equal to the number of representatives H of conjugacy classes of subgroups of G which have no normal subgroup of index p for an odd prime p. And that such a representative (except for the trivial subgroup) then has a normal subgroup of index 2. And on the other hand $\mathrm{rk}\,\Omega_2(G) - \mathrm{rk}\,\Omega(G) = r$ is greater or equal to the number of representatives H of conjugacy classes of subgroups of G which have a normal subgroup of index 2. This verifies the conjecture for all solvable groups G.

Does Feit-Thompson imply the conjecture for all finite groups G?

In general it seems that, the larger the group the larger the difference between the two quantities. This is illustrated by the following table, if compared with the table in section 3.

G	$\mathrm{rk}\,\Omega_2(G)$ $-\mathrm{rk}\,\Omega(G)$	G	$\mathrm{rk}\,\Omega_2(G)$ $-\mathrm{rk}\,\Omega(G)$	G	$\mathrm{rk}\,\Omega_2(G)$ $-\mathrm{rk}\,\Omega(G)$	G	$\mathrm{rk}\,\Omega_2(G)$ $-\mathrm{rk}\,\Omega(G)$
A_3	0	S_3	2	M_{11}	36	J_1	29
A_4	2	S_4	11	M_{12}	221	J_2	178
A_5	4	S_5	19	M_{22}	217		
A_6	14	S_6	82	M_{23}	243		
A_7	27	S_7	153	M_{24}	5512		
A_8	199	S_8	699				
A_9	305	S_9	1328				

Moreover, the conjecture has been verified for all groups of order less than 960.

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