

# On Subgroups of the Pentagon Group

CLAAS E. RÖVER

*Department of Mathematics*  
*National University of Ireland, Galway*  
*University Road, Galway*  
Claas.Roever@NUIGalway.ie

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This note presents a very short algebraic proof for a new result which implies the known fact that the fundamental group of a closed orientable surface is a subgroup of a graph group.

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Given a graph  $\Gamma$  with vertex set  $X$ , the graph group of  $\Gamma$ , denoted by  $G(\Gamma)$ , is the group generated by the elements of  $X$  subject to the relations  $xy = yx$  for each pair  $x$  and  $y$  of adjacent vertices in  $\Gamma$ . In the literature graph groups are also known as partially commutative or right angled Artin groups. It has been known for some time that certain surface groups are embeddable into graph groups [4, 1]. In [4] it was shown that the fundamental group of a closed orientable surfaces of genus  $1 + (n - 4)2^{n-3}$  embeds into the graph group of the  $n$ -gon for  $n \geq 4$ . Crisp and Wiest [1] deal with all surface groups which can possibly be embedded into a graph group and their embeddings are, in addition, quasi-isometric. The methods in these two papers are topological. The purpose of this note is to establish the following result which implies that the fundamental group of every closed orientable surface is embeddable into the graph group of the pentagon. This result appears to be new and its proof is fairly short and purely algebraic.

**Theorem 1** *Let  $P$  be the graph group of the pentagon whose vertices are labeled  $x_1, x_2, \dots, x_5$  in cyclic order. Put  $u = x_3x_1$ ,  $v = x_2x_4$  and let  $G$  be the subgroup of  $P$  generated by  $u$ ,  $v$  and  $x_5$ . Then  $G$  is isomorphic to the HNN-extension  $H = \langle U, V, T \mid [[U, V], T] = 1 \rangle$ , where  $[U, V] = U^{-1}V^{-1}UV$ .*

It is plain that the trivial group and the free abelian group of rank two are subgroups of  $P$ . Since the subgroup of  $H$  generated by  $U, V, U^T$  and  $V^T$  is isomorphic to the fundamental group of a closed orientable surface of genus two which, in turn, is covered by every closed orientable surface of genus at least two [3, section V.1], we have the following corollary.

**Corollary 2** *The fundamental group of any closed orientable surface is (isomorphic to) a subgroup of the graph group of the pentagon.*

Let us turn to the proof of the theorem. Write  $w = w(a, b, c)$  to indicate that  $w$  is a word over the letters  $a, b$  and  $c$  and their inverses. Then  $w(p, q, r)$  is the word obtained from  $w(a, b, c)$  by replacing each occurrence of  $a^\varepsilon, b^\varepsilon$  and  $c^\varepsilon$  by  $p^\varepsilon, q^\varepsilon$  and  $r^\varepsilon$  respectively for  $\varepsilon \in \{\pm 1\}$ . The following lemma is a very special case of a similar result about graph products of groups [2].

**Lemma 3** *A shortest representative of the element of  $P$  represented by a word  $w = w(x_1, x_2, x_3, x_4, x_5)$  can be obtained from  $w$  by a finite number of applications of reductions  $x_i^\varepsilon x_i^{-\varepsilon} \rightarrow \lambda$  and swaps  $x_i^\varepsilon x_j^\delta \rightarrow x_j^\delta x_i^\varepsilon$  where  $\varepsilon, \delta \in \{\pm 1\}$ ,  $|i - j| \in \{1, 4\}$ ,  $1 \leq i, j \leq 5$  and  $\lambda$  denotes the empty word.*

A sequence of words  $w_i(x_1, x_2, x_3, x_4, x_5)$ ,  $1 \leq i \leq k$ , in which  $w_{i+1}$  is obtained from  $w_i$ ,  $1 \leq i < k$ , by a reduction or a swap is called a *rewriting sequence* on  $w_1$ .

Let  $F$  be the subgroup of  $P$  which is generated by  $x_1$  and  $x_4$ . Then  $F$  is a free group of rank two, by Lemma 3, and there is a surjective homomorphism  $\pi : P \rightarrow F$ , induced by setting  $x_2 = x_3 = x_5 = 1$ . Let  $Q$  be the associated subgroup of the HNN-extension  $H$ , that is  $Q$  is generated by  $[U, V]$ . It is easy to check that  $[u, v] = [x_1, x_4]$ . Hence  $[[u, v], x_5] = 1$  in  $G$  and there is a surjective homomorphism  $\varphi : H \rightarrow G$  induced by setting  $U^\varphi = u, V^\varphi = v$  and  $T^\varphi = x_5$ .

Suppose  $w$  is an element of the kernel of  $\varphi$ . Then, up to conjugacy,  $w$  is represented by a word of the form  $w(U, V, T) = w_1 T^{n_1} w_2 T^{n_2} \dots w_l T^{n_l}$  satisfying either (i) or (ii) below.

- (i)  $l > 1$  and for  $1 \leq i \leq l$ , each  $w_i = w_i(U, V)$  is a non-empty freely reduced word and  $n_i \in \mathbb{Z} \setminus \{0\}$ .
- (ii)  $l = 1$  and  $w_1(U, V)$  is a freely reduced word such that  $n_1 \neq 0$  or  $w_1$  is non-empty, or both.

The *length* of  $w$  is the smallest  $l$  for which a conjugate of  $w$  has a representative satisfying one of the above conditions. It suffices to show that  $w$  is trivial in  $H$  which is done by induction on its length.

Clearly  $w^\varphi = w(u, v, x_5) = w(x_3x_1, x_2x_4, x_5)$ . Since  $w^\varphi = 1$ , there is some rewriting sequence on  $w^\varphi$  ending in a word not involving  $x_5^{\pm 1}$ , by Lemma 3. So  $\sum_{k=1}^l n_k = 0$ .

Thus, if  $l = 1$ , then  $w = w_1$ ,  $1 = w^\varphi = w_1(u, v)$  and hence  $1 = w^{\varphi\pi} = w_1(x_1, x_4)$ . Since  $F$  is the free group on  $\{x_1, x_4\}$  and  $w_1$  is freely reduced, it follows that  $w_1$  is the empty word. So  $w = 1$  in  $H$ .

Next assume that  $l > 1$ . It follows directly from Lemma 3 that the centraliser of  $x_5$  in  $P$  is the direct product of  $F$  and the cyclic subgroup generated by  $x_5$ . Since there is some rewriting sequence on  $w^\varphi$  which eliminates all occurrences of  $x_5^{\pm 1}$ , it follows that  $[x_5, w_i(u, v)] = 1$  for some  $i$ ,  $1 \leq i \leq l$ . The following lemma now completes the proof because it implies that  $w(U, V, T)$  is equal in  $H$  to  $w' = w_1T^{n_1} \cdots w_{n_{i-2}}T^{n_{i-2}}w_{i-1}T^{n_{i-1}+n_i}w_iw_{i+1}T^{n_{i+1}} \cdots w_lT^{n_l}$  whose length is less than the length of  $w$ , and hence  $1 = w' = w$  in  $H$ .

**Lemma 4** *If  $w = w(u, v)$  is a freely reduced word representing an element of  $F = \langle x_1, x_4 \rangle$ , then  $w = [u, v]^k$  for some  $k \in \mathbb{Z}$ .*

**Proof.** Let  $|w|_x$  denote the exponent sum of  $x$  in  $w$ . Observe that  $|w|_u = |w|_{x_1} = |w|_{x_3}$  and  $|w|_v = |w|_{x_2} = |w|_{x_4}$ . Since  $w$  can be reduced (by swaps and reductions) to a word which involves neither  $x_2^{\pm 1}$  nor  $x_3^{\pm 1}$ ,  $|w|_u = |w|_v = 0$ . So  $w$  is of the form  $u^{i_1}v^{j_1} \cdots u^{i_l}v^{j_l}$  with  $\sum_{k=1}^l i_k = \sum_{k=1}^l j_k = 0$  where  $i_k$  and  $j_k$  are non-zero integers except for  $i_1$  and  $j_l$  which may be zero. Observe that  $l > 1$ .

Since the restriction of  $\pi$  to the subgroup of  $G$  generated by  $u$  and  $v$  is an isomorphism and  $F$  is free, it follows that no rewriting sequence on  $w(x_3x_1, x_2x_4)$  can involve reductions  $x_j^\varepsilon x_j^{-\varepsilon}$  with  $j \in \{1, 4\}$ .

Suppose for a moment that, for some  $k$ ,  $i_k$  and  $i_{k+1}$  are of the same sign or that  $|i_k| > 1$ . Then  $w$ , or maybe  $w^{-1}$ , has a subword of the form  $x_3x_1(x_2x_4)^m \underline{x_3}x_1$  with  $m \in \mathbb{Z}$ . The previous paragraph and the fact that  $[x_1, x_3] \neq 1$  now imply that it is impossible to cancel the underlined  $x_3$  in any rewriting sequence on  $w$ . Similarly, considering  $x_2x_4(x_1x_3)^m \underline{x_2}x_4$ , shows that  $j_k$  and  $j_{k+1}$  must be of opposite sign and absolute value one for  $1 \leq k < l$ .

It follows that  $w$  or  $w^{-1}$  is of the form  $(u^{-\varepsilon}v^{-\delta}u^\varepsilon v^\delta)^m$  for some  $m \in \mathbb{Z}$ . Simple calculations reveal that  $\varepsilon = \delta = 1$ , as  $w$  is an element of  $F$ . This establishes the lemma, and hence the theorem.

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