$\mathcal{P}(\mathbb{R})$, ordered by homeomorphic embeddability, does not represent all posets of cardinality $2^{\mathfrak{c}}$

Robin W. Knight & Aisling E. McCluskey

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Abstract

We prove it to be consistent that there is a poset of cardinality $2^{\mathfrak{c}}$ which is not realizable in $\mathcal{P}(\mathbb{R})$, ordered by homeomorphic embeddability. This addresses and answers resolutely (and in the negative) the open question of whether there is a ZFC theorem that all posets of cardinality $2^{\mathfrak{c}}$ can be represented by subspaces of the real line ordered by homeomorphic embeddability. This question arises from the pioneering work of Banach, Kuratowski and Sierpiński in the area and this result complements the recent work of [9], thus providing a proof of independence.

1   Introduction

The ordering by embeddability of topological spaces, although a fundamental notion in topology, has been remarkably little understood for some years. This ordering is that introduced into a family of topological spaces by writing $X \hookrightarrow Y$ whenever X is homeomorphic to a subspace of Y. Its subtlety and relative intractability are well illustrated by the problem of recognising which order-types are those of collections of subspaces of the real line $\mathbb{R}$ (see [4], [5], [6], [7], [8]). A partially-ordered set (poset) P is *realized* (or *realizable*) within a family $\mathcal{F}$ of topological spaces whenever there is an injection $\theta : P \to \mathcal{F}$ for which $p \le q$ if and only if $\theta(p) \hookrightarrow \theta(q)$. Discussion of realizability in the powerset $\mathcal{P}(\mathbb{R})$ can be traced back to Banach, Kuratowski and Sierpiński ([2], [3], [11]), whose work on the extensibility of continuous maps over G_δ subsets (in the context of Polish spaces) revealed *inter alia* that it is possible to realize, within $\mathcal{P}(\mathbb{R})$, (i) the antichain of cardinality $2^{\mathfrak{c}}$ [2, p. 205] and (ii) the ordinal $\mathfrak{c}^+$ [3, p. 199]. Renewed interest in the problem was initiated in [4] in which it was shown that every poset of cardinality $\mathfrak{c}$ or less can be realized within $\mathcal{P}(\mathbb{R})$. Till now, the question of precisely which posets of cardinalities exceeding $\mathfrak{c}$ can be so realized had been unresolved. Indeed, the question ultimately revealed itself to be set-theoretic in nature. Accordingly, this paper seeks to answer this question by forcing to construct a poset of cardinality $2^{\mathfrak{c}}$ which cannot be realized within $\mathcal{P}(\mathbb{R})$. This, together with [9], then provides a complete picture of the order-theoretic structure of $\mathcal{P}(\mathbb{R})$, namely that the statement that all posets of cardinality $2^{\mathfrak{c}}$ can be realized within $\mathcal{P}(\mathbb{R})$ is independent of ZFC.

2 Forcing argument

All forcing-related notation in this section is taken from [1].

Theorem 2.1 *It is consistent with ZFC that there exists a partially ordered set of cardinality $2^{\mathfrak{c}}$ which is not realizable in $\mathcal{P}(\mathbb{R})$.*

Proof. We assume that in the ground model $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_3$, and $2^{\aleph_2} = \aleph_4$. We will construct a bipartite graph, and construct the partially-ordered set from it that cannot be realized in $\mathcal{P}(\mathbb{R})$. Our graph will have a set of vertices

$$V = \omega_2 \times \{0\} \cup \omega_3 \times \{1\},$$

and the edge relation E will be a subset of $\big(\omega_2 \times \{0\}\big) \times \big(\omega_3 \times \{1\}\big)$. We will code E by a relation $F \subseteq \omega_2 \times \omega_3$. The partially-ordered set will then be V, ordered so that $\langle \alpha, i \rangle \leq \langle \beta, j \rangle$ if and only if either $\langle \alpha, i \rangle = \langle \beta, j \rangle$, or $i = 0$, $j = 1$, and $\big\langle \langle \alpha, i \rangle, \langle \beta, j \rangle \big\rangle \in E$.

We create F by forcing. The forcing poset will be

$$\mathbb{P} = \mathrm{Fn}(\omega_2 \times \omega_3, 2, \omega_3)$$

Since $\mathbb{P}$ is $\mathfrak{c}^+$-closed, forcing with $\mathbb{P}$ creates no new reals and no new sets of reals. Let $\dot{F}$ be a name for the generic subset of $\omega_2 \times \omega_3$ that we have created by forcing over $\mathbb{P}$. We show that,in the generic extension, the associated partial order, as defined above, is not realizable, by considering possible maps $\phi : V \to \mathcal{P}(\mathbb{R})$; we aim to show that it is impossible that $\phi(\alpha, 0)$ is (homeomorphically) embeddable in $\phi(\beta, 1)$ iff $\langle \alpha, \beta \rangle \in F$.

In other words, we show that for every name $\dot{\phi}$ for such a map, for all $p \in \mathbb{P}$, there exists $q \leq p$ such that, for some α, β, either

$$q \Vdash \check{}\langle \alpha, \beta \rangle \in \dot{F} \ \wedge \ \dot{\phi}(\check{\alpha}, \check{0}) \not\hookrightarrow \dot{\phi}(\check{\beta}, \check{1}),$$

or

$$q \Vdash \check{}\langle \alpha, \beta \rangle \notin \dot{F} \ \wedge \ \dot{\phi}(\check{\alpha}, \check{0}) \hookrightarrow \dot{\phi}(\check{\beta}, \check{1}).$$

So, given $\dot{\phi}$, and p, we first exploit ω_3-closure of $\mathbb{P}$ to extend p to a condition $r \leq p$ such that for all $\alpha < \omega_2$, there exists $A_\alpha \in \mathcal{P}(\mathbb{R})$ such that

$$r \Vdash \dot{\phi}(\check{\alpha}, \check{0}) = \check{A}_\alpha.$$

For each subset C of $\mathbb{R}$, let

$$D_C = \{\alpha : A_\alpha \hookrightarrow C\}.$$

Now $\mathcal{P}(\mathbb{R})$ has only $\aleph_3$ elements, while $2^{\aleph_2} = \aleph_4$. Therefore there exists a subset Δ of ω_2 such that for all C, $\Delta \neq D_C$.

Now pick $\beta \in \omega_3$ such that for all $\alpha \in \omega_2$, $\langle \alpha, \beta \rangle \notin \mathrm{dom}\, r$, which we can do since r has cardinality at most $\aleph_2$.

Define $s \leq r$ so that if $\alpha \in \Delta$, $s(\alpha, \beta) = 1$, and if $\alpha \notin \Delta$, then $s(\alpha, \beta) = 0$.

Now find $q \leq s$ such that for some $C \subseteq \mathbb{R}$,

$$q \Vdash \dot{\phi}(\check{\beta}, \check{1}) = \check{C}.$$

Now, recall that $\Delta \neq D_C$. Therefore, either there exists $\alpha \in \Delta$ such that $\alpha \notin D_C$, or there exists $\alpha \notin \Delta$ such that $\alpha \in D_C$.

In the first case,

$$q \Vdash \check{\ }\langle \alpha, \beta \rangle \in \dot{F} \ \wedge \ \dot{\phi}(\check{\alpha}, \check{0}) \not\hookrightarrow \dot{\phi}(\check{\beta}, \check{1}),$$

and in the second case,

$$q \Vdash \check{\ }\langle \alpha, \beta \rangle \notin \dot{F} \ \wedge \ \dot{\phi}(\check{\alpha}, \check{0}) \hookrightarrow \dot{\phi}(\check{\beta}, \check{1}),$$

and the proof is complete.
$\square$

3   Future Hopes

In its most general form, the question of determining, given a poset P and a topological space X, if there are subspaces of X whose embeddability ordering precisely matches that of the ordering within P is relatively unexplored and deserving of investigation. We now know that for a space as familiar as $\mathbb{R}$, the order-theoretic structure of its powerset $\mathcal{P}(\mathbb{R})$ is inherently set-theoretic and ultimately cannot be determined in ZFC. An obvious question arises as to how much (or how little) of the topological nature of $\mathbb{R}$ has influenced this outcome. The argument presented would seem to work for any space of cardinality $\mathfrak{c}$. What about spaces of higher cardinality? Further, for such poset representations within $\mathcal{P}(\mathbb{R})$ as are possible, is it possible to restrict the representative subspaces to some 'nice' family of subsets of $\mathbb{R}$? In [10], further work along these lines is presented.

More generally, the relation of embeddability is of interest in that in studying it, we are studying the structure of the category of topological spaces, or rather, of those spaces which are homeomorphic to subsets of $\mathbb{R}$. Since all subspaces of $\mathbb{R}$ which include an interval are equivalent under the relation of embeddability, what we are essentially studying is the structure of the category of all zero-dimensional separable metric spaces. It would be natural, therefore, to broaden our study to include other categories of topological spaces, such as zero-dimensional spaces of weight κ, or spaces of cardinality κ, and ask what can be said about these.

4   Acknowledgement

References

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