Parameter uniform approximations for time-dependent reaction-diffusion problems

Torsten Linß* and Niall Madden†

June 2006

Abstract

Using discrete Green's functions techniques, we present a classification of fitted mesh methods for time-dependent reaction diffusion problems, based on the analyses of Linß [4] for the analogous steady-state problem and of Kopteva [3] of time-dependent convection-diffusion problems.

As examples of how to apply the analysis, we derive error estimates for the fitted meshes of Shishkin and Bakhvalov, and provide supporting numerical results.

1 Introduction

We consider the problem of computing a satisfactory numerical solution to a time dependent singularly perturbed reaction-diffusion equation using a standard finite different method. The problem is

$$u_t + \mathcal{L}_\varepsilon u = f \qquad \text{in} \quad (0,1) \times (0,T], \tag{1a}$$

where $\mathcal{L}_\varepsilon v := -\varepsilon^2 v_{xx} + rv$, $r : (0,1) \to \mathbb{R}$, subject to boundary conditions

$$u(0,t) = \gamma_0(t),\ u(1,t) = \gamma_1(t) \quad \text{in} \quad (0,T], \tag{1b}$$

and initial condition

$$u(\cdot,0) = u^0 \qquad \text{in} \quad (0,1). \tag{1c}$$

Solutions to (1) typically exhibit layers: narrow regions in which derivatives of the solution are large. More precisely, application of the technique from [6, § 3] gives the bounds

$$\left|\partial_x^\ell \partial_t^k u(x,t)\right| \le C\left\{\varepsilon^{\min\{0,2-\ell\}} + \varepsilon^{-\ell}\mathrm{e}^{-\varrho x/\varepsilon} + \varepsilon^{-\ell}\mathrm{e}^{-\varrho(1-x)/\varepsilon}\right\} \quad \text{for } \ell = 0,\dots,4,\ k = 0,1,2 \tag{2}$$

if the data is sufficiently smooth which will be assumed throughout. A satisfactory numerical scheme for problems of this type should resolve the boundary layers. To achieve this with a mesh that is uniform in the spatial dimension, one must choose the number of mesh points to be proportional to $1/\varepsilon$. That is not computationally feasible, so specially constructed nonuniform meshes are necessary. These meshes should also yield methods that are *parameter robust*: the error in the computed solution for a fix number of mesh points should be essentially independent of ε.

*Institut für Numerische Mathematik, Technische Universität Dresden, D-01062 Dresden, Germany; email: Torsten.Linss@tu-dresden.de

†Department of Mathematics, National University of Ireland, Galway, Ireland; email: Niall.Madden@nuigalway.ie. The work of this author was supported by the NUI Galway Millennium Research Fund 112/2005.

The piecewise uniform "Shishkin" meshes [5, Chap. 6] and the graded meshes of Bakhvalov [2] have been shown to lead to important examples of layer-resolving, parameter uniform fitted-mesh methods for the the time-independent analog of (1). Furthermore, it has been shown [6] that Shishkin meshes can be extended easily to yield parameter robust solutions to (1). Proving this requires a special decomposition of the solution into purely layer and regular parts, and the use of special barrier-function based techniques to prove the optimal order of convergence of the method.

In an alternative approach based discrete Green's functions, Linß [4] provides a general classification of fitted meshes for the time-independent problem that avoid either solution decomposition and the use of special barrier functions. The approach has the important property that Shishkin and Bakhvalov meshes can be analysed in the same framework. In §3 we give a useful simplification of the analysis of [4], and in §4 extend it to the time-dependent problem (1).

This allows us in §5.1 to deduce the main result of [6]. The analysis for Bakhvalov meshes is presented in §5.2. Supporting numerical results are given in §6.

Notation. In space we denote an arbitrary mesh as $\omega_x^N : 0 = x_0 < x_1 < \cdots < x_N = 1$ and in time $\omega_t^K : 0 = t_0 < t_1 < \cdots < t_K = T$. Let $h_i := x_i - x_{i-1}$ be the mesh diameter in the spatial dimension, and $\hbar_i := (h_i + h_{i+1})/2$. Time step sizes are $\tau_j := t_j - t_{j-1}$, with $\tau := \max_{j=1,\dots,K} \tau_j$.

The approximation to (1) is computed on a mesh $\omega^{N,K}$ that is the tensor product of the one-dimensional meshes ω_x^N and ω_t^K.

In §3 we consider just a stationary problem, and so use g_i to denote the value one-dimensional mesh function at x_i. For the two-dimensional time-dependent problem we let g_i^j denote the value of mesh function g at the point (x_i, t_j). In §4 we use g^j to denote one-dimensional mesh function at time-step j.

We use C to denote a generic constant that is independent of ε, N and K.

2 Discretization

We discretize (1) using central differences in space and backward differences in time:

$$v_{\bar{x};i} := \frac{v_i - v_{i-1}}{h_i}, \quad v_{\hat{x};i} := \frac{v_{i+1} - v_i}{\hbar_i} \quad \text{and} \quad v_{\bar{t}}^j := \frac{v^j - v^{j-1}}{\tau_j}.$$

The operator $\mathcal{L}_\varepsilon$ is approximated by

$$[L_\varepsilon v]_i := -\varepsilon^2 v_{\bar{x}\hat{x};i} + r_i v_i.$$

The numerical approximation U of (1) is the solution the linear different equation

$$[U_{\bar{t}} + L_\varepsilon U]_i^j = f_i^j \quad \text{for} \quad i = 1, \dots, N-1, \;\; j = 1, \dots, K, \tag{3a}$$

with the boundary and initial conditions discretized by

$$U_0^j = \gamma_0^j, \; U_N^j = \gamma_1^j \quad \text{for} \quad j = 0, \dots, K, \tag{3b}$$

and

$$U_i^0 = u_i^0 \quad \text{for} \quad i = 1, \dots, N-1. \tag{3c}$$

3 The stationary problem

Consider the difference scheme

$$[L_\varepsilon U]_i := f_i \quad \text{for} \quad i = 1, \dots, N-1, \;\; U_0 = \gamma_0, \; U_N = \gamma_1 \tag{4}$$

as a discretization of the stationary reaction-diffusion problem

$$\mathcal{L}_\varepsilon u = -\varepsilon^2 u'' + ru = f \quad \text{in } (0,1), \quad u(0)=\gamma_0,\ u(1)=\gamma_1 \tag{5}$$

with $r \geq \varrho^2$, $\varrho > 0$. This type of problem has been studied in the context of layer-adapted meshes in a number of publications, e.g., [5, 1, 4]. Here we shall recall the stability results from [1] and present a modification of the error analysis in [4]. Both are important ingredients needed to study the time-dependent problem (1).

Properties of the exact solution. For the solution u of (5) we have the derivative bounds

$$\left|u^{(l)}(x)\right| \leq C\left\{\varepsilon^{\min\{0,2-\ell\}} + \varepsilon^{-\ell}\mathrm{e}^{-\varrho x/\varepsilon} + \varepsilon^{-\ell}\mathrm{e}^{-\varrho(1-x)/\varepsilon}\right\} \quad \text{for } \ell = 0,\dots,4, \tag{6}$$

see, [5, Lemma 6.1].

Discrete Green's function. Let δ_j^i denote the usual Kronecker delta: it is the mesh function defined on ω_x^N such that for all $0 \leq i, j \leq N$,

$$\delta_j^i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Then the discrete Green's function G^i associated with L_ε and the mesh node x_i is the solution to

$$\left[L_\varepsilon G^i\right]_j := \hbar_j^{-1}\delta_j^i \quad \text{for} \quad j = 1,\dots,N-1, \qquad G_0^i = G_N^i = 0.$$

Any mesh function $v = (v_0,\dots,v_N)$ with $v_0 = v_N = 0$ can be represented as

$$v_i = \sum_{k=1}^{N-1} \hbar_k G_k^i \left[L_\varepsilon v\right]_k \quad \text{for } i = 1,\dots,N-1.$$

Therefore the set $\{G^i\}$ forms a useful basis for expressing solutions to (4), and important properties of the discrete operator can derived by studying the associated Green's functions.

Stability of L_ε. This representation requires G^i to be defined at the mesh nodes only, however it is convenient to interpret G as a piecewise linear function on the mesh ω_x^N. Then we have the estimates [1, §2]

$$\int_0^1 G^i(\xi)\,d\xi = \sum_{k=1}^{N-1} \hbar_k G_k^i \leq \frac{1}{\varrho^2}, \tag{7a}$$

and

$$\int_0^1 \left|\left(G^i(\xi)\right)'\right|\,d\xi = 2G_i^i \leq \frac{2}{\varepsilon\varrho}. \tag{7b}$$

An immediate consequence of (7a) is the stability inequality

$$\|v\|_\infty := \max_{i=1,\dots,N-1} |v_i| \leq \frac{1}{\varrho^2}\|L_\varepsilon v\|_\infty, \tag{8}$$

which hold true for arbitrary mesh functions v with $v_0 = v_N = 0$.

Interpolation. Let ψ^I denote the piecewise linear interpolant to ψ on the given mesh ω_x^N. The interpolation error can be written as

$$\left(\psi - \psi^I\right)(x) = \frac{1}{h_i}\int_{x_{i-1}}^{x_i}\int_{x_{i-1}}^{x}\int_{\xi}^{s}\psi''(t)\,dt\,d\xi\,ds, \qquad \text{for } x \in [x_{i-1}, x_i]. \tag{9}$$

For triple integrals of this structure we have the two bounds

$$\frac{1}{h_i}\left|\int_{x_{i-1}}^{x_i}\int_{x_{i-1}}^{x}\int_{\xi}^{s}\chi(t)\,dt\,d\xi\,ds\right| \le \int_{x_{i-1}}^{x_i}\int_{\xi}^{x_i}|\chi(t)|\,dt\,d\xi \tag{10a}$$

and

$$\frac{1}{h_i}\left|\int_{x_{i-1}}^{x_i}\int_{x_{i-1}}^{x}\int_{\xi}^{s}\chi(t)\,dt\,d\xi\,ds\right| \le \int_{x_{i-1}}^{x_i}\int_{x_{i-1}}^{s}|\chi(t)|\,dt\,ds. \tag{10b}$$

These integrals can be further bounded using the following inequalities. Let $\chi : [a,b] \to \mathbb{R}$ be any function with $\chi \ge 0$ on (a,b). Then

$$\int_a^b\int_\xi^b \chi(t)\,dt\,d\xi \le \frac{1}{2}\left\{\int_a^b \chi(t)^{1/2}\,dt\right\}^2 \qquad \text{if } \chi \text{ is monotonically decreasing,} \tag{11a}$$

and

$$\int_a^b\int_a^s \chi(t)\,dt\,ds \le \frac{1}{2}\left\{\int_a^b \chi(t)^{1/2}\,dt\right\}^2 \qquad \text{if } \chi \text{ is monotonically increasing,} \tag{11b}$$

which can be verified by considering the left and right-hand sides as functions of the upper integration limit.

Error analysis. By means the Green's functions as a basis, the error at the mesh node x_i can be written as

$$\begin{aligned}
(u-U)_i &= \sum_{k=1}^{N-1}\hbar_k G_k^i\left[L_\varepsilon(u-U)\right]_k = \sum_{k=1}^{N-1}\hbar_k G^i\left[L_\varepsilon u - \mathcal{L}_\varepsilon\right]_k \\
&= \varepsilon^2\sum_{k=0}^{N-1}\frac{u_{k+1}-u_k}{h_{k+1}}\left(G_{k+1}^i - G^i\right) + \varepsilon^2\sum_{k=1}^{N-1}\hbar_k G^i u_k'' \\
&= \varepsilon^2\int_0^1 u'(x)(G^i)'(x)dx + \varepsilon^2\sum_{k=1}^{N-1}\hbar_k G^i u_k'' \qquad \text{(because } (G^i)' \text{ is piecewise constant)} \\
&= -\varepsilon^2\int_0^1 u''(x)G(x)dx + \varepsilon^2\sum_{k=1}^{N-1}\hbar_k G^i u_k''.
\end{aligned}$$

Thus, with $\varphi := -\varepsilon^2 u''$,

$$(u-U)_i = \int_0^1\left\{(G^i\varphi)(x) - (G^i\varphi)^I(x)\right\}dx. \tag{12}$$

Using (9), we have the representation

$$\begin{aligned}
(G^i\varphi)(x) - (G^i\varphi)^I(x) = 2G_{\bar{x},k}^i\frac{1}{h_k}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{x}\int_{\xi}^{s}\varphi'(t)\,dt\,d\xi\,ds \\
+ \frac{1}{h_k}\int_{x_{k-1}}^{x_k}\int_{x_{k-1}}^{x}\int_{\xi}^{s}\left(G^i\varphi''\right)(t)\,dt\,d\xi\,ds \quad \text{for } x \in [x_{k-1}, x_k].
\end{aligned} \tag{13}$$

Next we like to apply (10) and (11) to the triple integrals on the right-hand side. Therefore we split φ' into two parts that can be bounded by monotone functions—one decreasing and the other increasing. Set

$$\varphi_D(x) := \begin{cases} \varphi'(x) & \text{for } x \le \frac{1}{2}, \\ 0 & \text{for } x \ge \frac{1}{2} \end{cases} \qquad \text{and} \qquad \varphi_I(x) := \begin{cases} 0 & \text{for } x \le \frac{1}{2}, \\ \varphi'(x) & \text{for } x \ge \frac{1}{2}. \end{cases}$$

Clearly (6) with $\ell = 3$ yields

$$\varepsilon^{-1} |\varphi_D(x)| \le C \left\{ 1 + \varepsilon^{-2} \mathrm{e}^{-\varrho x/\varepsilon} \right\} \quad \text{and} \quad \varepsilon^{-1} |\varphi_I(x)| \le C \left\{ 1 + \varepsilon^{-2} \mathrm{e}^{-\varrho(1-x)/\varepsilon} \right\}.$$

Hence, using (10) and (11), we obtain

$$\frac{1}{\varepsilon h_k} \left| \int_{x_{k-1}}^{x_k} \int_{x_{k-1}}^{x} \int_{\xi}^{s} \varphi'(t)\, dt\, d\xi\, ds \right| \le C \vartheta \left(\omega_x^N\right)^2, \tag{14}$$

where

$$\vartheta\left(\omega_x^N\right) := \max_{k=1,\dots,N} \int_{x_{k-1}}^{x_k} \left(1 + \varepsilon^{-1} \mathrm{e}^{-\varrho t/2\varepsilon} + \varepsilon^{-1} \mathrm{e}^{-\varrho(1-t)/2\varepsilon} \right) dt. \tag{15}$$

The second integral in (13) is bounded in a similar manner. Set

$$\bar\varphi_D(x) := \begin{cases} \varphi''(x) & \text{for } x \le \frac{1}{2}, \\ 0 & \text{for } x \ge \frac{1}{2} \end{cases} \qquad \text{and} \qquad \bar\varphi_I(x) := \begin{cases} 0 & \text{for } x \le \frac{1}{2}, \\ \varphi''(x) & \text{for } x \ge \frac{1}{2}. \end{cases}$$

Then (6) with $\ell = 4$ gives, for $x \in [x_{k-1}, x_k]$,

$$\left| \left(G^i \bar\varphi_D \right)(x) \right| \le C \left(G_{k-1}^i + G_k^i \right) \left\{ 1 + \varepsilon^{-2} \mathrm{e}^{-\varrho x/\varepsilon} \right\}$$

and

$$\left| \left(G^i \bar\varphi_I(x) \right) \right| \le C \left(G_{k-1}^i + G_k^i \right) \left\{ 1 + \varepsilon^{-2} \mathrm{e}^{-\varrho(1-x)/\varepsilon} \right\},$$

since G^i is piecewise linear and positive. Use (10) and (11) once again in order to get

$$\frac{1}{h_k} \left| \int_{x_{k-1}}^{x_k} \int_{x_{k-1}}^{x} \int_{\xi}^{s} G^i \varphi''(t)\, dt\, d\xi\, ds \right| \le C \left(G_{k-1}^i + G_k^i \right) \vartheta \left(\omega_x^N\right)^2 \quad \text{for } x \in [x_{k-1}, x_k]. \tag{16}$$

Applying (14) and (16) to (13), we obtain

$$\left| (G^i \varphi)(s) - (G^i \varphi)^I(s) \right| \le C \vartheta \left(\omega_x^N\right)^2 \left(\varepsilon G_{\bar x,k}^i + G_{k-1}^i + G_k^i \right). \tag{17}$$

Finally, integrate over $[0, 1]$. We get

$$\left| \int_0^1 \left\{ (G^i \varphi)(x) - (G^i \varphi)^I(x) \right\} dx \right| \le C \vartheta \left(\omega_x^N\right)^2 \left(\varepsilon \int_0^1 G_{\bar x}^i(x)\, dx + \int_0^1 G^i(x)\, dx \right).$$

Hence

$$\|u - U\|_\infty \le C \vartheta \left(\omega_x^N\right)^2,$$

by (7) and (12). We have recovered [4, Theorem 2], however the representation seems to be more direct and to the point.

4 The time-dependent problem

After having studied properties of the spatial discretization, we can now proceed with the analysis of (3). Our analysis follows and modifies the technique in [3] for convection-diffusion problems.

Stability of the time discretization.

Lemma 1. *Suppose y satisfies*

$$\begin{aligned}[y_{\bar t} + L_\varepsilon y]_i^j = g_i^j \quad &\textit{for} \quad i = 1,\dots,N-1, \quad j = 1,\dots,K,\\ y_0^j = y_N^j = 0 \quad &\textit{for} \quad j = 0,\dots,K.\end{aligned}$$

Then

$$\left\|y^j\right\|_\infty \le \left\|y^0\right\|_\infty + \sum_{\nu=1}^{j} \tau_\nu \left\|g^\nu\right\|_\infty \quad \textit{for} \quad j = 0,\dots,K.$$

Proof. For $\nu = 1,\dots,K$ we have

$$-\varepsilon^2 \tau_\nu y^\nu_{\bar x \hat x} + (1 + \tau_\nu r)\, y^\nu = y^{\nu-1} + \tau_\nu g^\nu.$$

Application of (8) with ϱ^2 replaced by $1 + \tau_\nu \varrho^2$ yields

$$\left\|y^\nu\right\|_\infty \le \frac{1}{1+\tau_\nu \varrho^2} \left\|y^{\nu-1} + \tau_\nu g^\nu\right\|_\infty \le \left\|y^{\nu-1}\right\|_\infty + \tau_\nu \left\|g^\nu\right\|_\infty .$$

The proposition of the lemma follows by induction for $\nu = 1,\dots,j$. □

An immediate consequence of Lemma 1 is the inequality

$$\max_{j=1,\dots,K} \left\|y^j\right\|_\infty \le \left\|y^0\right\|_\infty + T \max_{j=1,\dots,K} \left\|g^j\right\|_\infty . \tag{18}$$

Error analysis. Let $\eta = u - U$ denote the error of the difference scheme (3). Then the truncation error can be represented as

$$\left[\eta_{\bar t} + L_\varepsilon \eta\right]_i^j = \psi_{1;i}^j + \psi_{2;i}^j$$

where

$$\psi_{1;i}^j = [L_\varepsilon u]_i^j - (\mathcal{L}_\varepsilon u)_i^j \qquad \text{and} \qquad \psi_{2;i}^j = u_{\bar t;i}^j - u_{t;i}^j$$

are the truncation errors for the space and time discretization, respectively.

With this splitting of the truncation error we can decompose the error η as $\eta = w + v$ where w and v solve

$$[L_\varepsilon w]_i^j = \psi_{1;i}^j, \quad i = 1,\dots,N-1, \quad w_0^j = w_N^j = 0, \quad j = 0,\dots,K, \tag{19a}$$

and

$$\begin{aligned}[v_{\bar t} + L_\varepsilon v]_i^j &= \psi_{2;i}^j - w_{\bar t;i}^j, \quad i = 1,\dots,N-1, \quad v_0^j = v_N^j = 0, \quad j = 1,\dots,K,\\ v_i^0 &= -w_i^0, \quad i = 0,\dots,N.\end{aligned} \tag{19b}$$

To bound w, note that (19a) is a sequence of stationary problems like those considered in Section 3 and can be bounded using the technique from that section and (2) with $\ell = 3, 4$:

$$\max_{j=0,\dots,N} \left\|w^j\right\|_\infty \le C\vartheta \left(\omega_x^N\right)^2 . \tag{20}$$

where $\vartheta\left(\omega_x^N\right)$ is as defined in (15).

Apply (18) to (19b). We get

$$\begin{aligned}\max_{j=0,\ldots,N}\left\|v^j\right\|_\infty &\le \left\|w^0\right\|_\infty + T\max_{j=1,\ldots,N}\left(\left\|u_{\bar t}^j - u_t^j\right\|_\infty + \left\|w_{\bar t}^j\right\|_\infty\right)\\ &\le C\vartheta\left(\omega_x^N\right)^2 + C\tau + T\max_{j=1,\ldots,N}\left\|w_{\bar t}^j\right\|_\infty,\end{aligned}$$

by (20), a Taylor expansion and (2) for $k = 2$, $\ell = 0$.

Finally, note that $w_{\bar t}^j$ solves

$$[L_\varepsilon w_{\bar t}]_i^j = \psi_{1;\bar t;i}^j, \quad i = 1,\ldots,N-1, \quad w_0^j = w_N^j = 0, \quad j = 1,\ldots,K.$$

This is again a sequence of stationary problems to which the technique from Section 3 can be applied. However, this time (2) with $k = 1$ and $\ell = 3, 4$ has to be used. We get

$$\max_{j=0,\ldots,N}\left\|v^j\right\|_\infty \le C\vartheta\left(\omega_x^N\right)^2 + C\tau.$$

Recalling (20), we obtain our final convergence result.

Theorem 1. *The maximum nodal error of the finite difference approximation* (3) *satisfies*

$$\max_{j=0,\ldots,K}\left\|u^j - U^j\right\|_\infty \le C\left(\vartheta\left(\omega_x^N\right)^2 + \tau\right).$$

5 Layer-adapted meshes

We now employ the result of Theorem 1 to analyse the discretization (3) on two standard layer-adapted meshes.

5.1 Shishkin meshes

Shishkin meshes [5, 7] are frequently studied because of their simplicity—they are piecewise uniform and so very easy to implement. We describe a possible construction for problem (1). Let $q \in (0, 1/2)$ and $\sigma > 0$ be mesh parameters. We set

$$\lambda = \min\left\{q, \frac{\sigma\varepsilon}{\varrho}\ln N\right\} \tag{21}$$

Assuming that qN is an integer, we construct ω_x^N by uniformly dividing each of the intervals $[0, \lambda]$ and $[1-\lambda, 1]$ into qN subintervals, while $[\lambda, 1-\lambda]$ is divided into $(1-2q)N$ subintervals.

For the mesh constructed this way we have [4]

$$\vartheta\left(\omega_x^N\right) \le C\left\{N^{-\sigma/2} + N^{-1}\ln N\right\}.$$

Therefore Theorem 1 yields the uniform error bound

$$\max_{j=0,\ldots,K}\left\|u^j - U^j\right\|_\infty \le C\left(N^{-2}\ln^2 N + \tau\right) \quad \text{if} \quad \sigma \ge 2. \tag{22}$$

5.2 Bakhvalov meshes

The graded Bakhvalov meshes [2] are superior to Shishkin meshes in the sense that they yield numerical solutions that contain smaller errors and have higher rates of convergence. However they are less popular in the literature because more advanced techniques are required in their analysis.

For problem (1) the Bakhvalov mesh may be regarded as generated by equidistributing the function

$$M_{Ba}(x) = \max\left\{1, \kappa\varepsilon^{-1}\mathrm{e}^{-\varrho x/\varepsilon\sigma}, \kappa\varepsilon^{-1}\mathrm{e}^{-\varrho(1-x)/\varepsilon\sigma}\right\},$$

with positive constants κ and σ, i. e., the mesh points x_i are chosen such that

$$\int_{x_{i-1}}^{x_i} M_{Ba}(x)dx = \frac{1}{N}\int_0^1 M_{Ba}(x)dx.$$

The parameter κ determines the number of mesh points used to resolve the layers, while σ determines the grading of the mesh inside them. Note also, that for $\varepsilon \ll 1$, the mesh is graded in the regions $[0, \lambda]$ and $[1-\lambda, 1]$, and piecewise uniform in $[\lambda, 1-\lambda]$ where $\lambda = \sigma(\varepsilon/\varrho)\ln(\kappa/\varepsilon)$. Compare with (21) above.

Clearly we have

$$1 + \varepsilon^{-1}\mathrm{e}^{-\varrho x/2\varepsilon} + \varepsilon^{-1}\mathrm{e}^{-\varrho(1-x)/2\varepsilon} \le CM_{Ba}(x) \quad \text{for} \quad \sigma \ge 2$$

and $\int_0^1 M_{Ba}(x)dx \le C$. Thus $\vartheta\left(\omega_x^N\right) \le CN^{-1}$. Then using Theorem 1, we can conclude

$$\max_{j=0,\dots,K}\left\|u^j - U^j\right\|_\infty \le C\left(N^{-2} + \tau\right) \quad \text{if} \quad \sigma \ge 2. \tag{23}$$

6 Numerical results

We consider the following example of (1)

$$u_t - \varepsilon^2 u_{xx}(x,t) + \sqrt{x+1}\,u(x,t) = 1 \quad (0,1)\times(0,1], \tag{24a}$$

with

$$u(0,s) = u(1,s) = u(s,0) \quad \text{ for } 0 \le s \le 1. \tag{24b}$$

The exact solution to problem (24) is not available, so we estimate the accuracy of a numerical solution by comparing it to the numerical solution computed on a much finer mesh.

As described in §2, the approximation is obtained on a mesh $\omega^{N,K}$ that is the tensor product of the one-dimensional meshes ω_x^N and ω_t^K. Therefore the mesh has N and K intervals on the x- and t-axes respectively. For a given ε, let $U_\varepsilon^{N,K}$ be the approximate solution to (24) computed on $\omega^{N,K}$.

Then bisect twice each interval of ω_x^N and ω_t^K and take their tensor product to obtain a mesh $\tilde{\omega}^{4N,4K}$ that has 16 times as many subintervals as $\omega^{N,K}$ Let $\tilde{U}_\varepsilon^{4N,4K}$ denote the solution computed on this mesh.

We then estimate the error for a given ε, N and K as

$$\eta_\varepsilon^{N,K} := \left\|U_\varepsilon^{N,K} - \tilde{U}_\varepsilon^{4N,4K}\right\|_{\omega^{N,K}}.$$

We also estimate the rates of convergence in the usual way:

$$r_\varepsilon^{N,K} = \log_2\left(\eta^N/\eta^{2N}\right).$$

6.1 A Shishkin mesh

We present the numerical results for Example (24) obtained on the Shishkin mesh. We take $q = 1/4$, $\sigma = 2$ and $\varrho = 1$ and construct ω_x^N as described in §5.1. The time mesh ω_t^K is uniform.

First we verify that the error is essentially independent of ε. Table 1 contains the numerical results for various values of ε and $N = K$. Alternate rows show the error for a given ε, and the rate

ε^2	$N=2^4$	$N=2^5$	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$	$N=2^{10}$
1	8.08e-03	4.61e-03	2.51e-03	1.31e-03	6.69e-04	3.38e-04	1.70e-04
$r_\varepsilon^{N,K}$	0.81	0.88	0.94	0.97	0.98	0.99	
1e-02	1.37e-02	5.73e-03	2.59e-03	1.23e-03	6.03e-04	2.98e-04	1.48e-04
$r_\varepsilon^{N,K}$	1.25	1.15	1.07	1.03	1.02	1.01	
1e-04	2.97e-02	1.49e-02	6.21e-03	2.51e-03	1.00e-03	4.16e-04	1.81e-04
$r_\varepsilon^{N,K}$	1.00	1.26	1.31	1.32	1.27	1.20	
1e-06	2.98e-02	1.49e-02	6.22e-03	2.51e-03	1.00e-03	4.16e-04	1.81e-04
$r_\varepsilon^{N,K}$	1.00	1.26	1.31	1.32	1.27	1.20	
1e-08	2.98e-02	1.49e-02	6.22e-03	2.51e-03	1.00e-03	4.16e-04	1.81e-04
$r_\varepsilon^{N,K}$	1.00	1.26	1.31	1.32	1.27	1.20	
1e-10	2.98e-02	1.49e-02	6.22e-03	2.51e-03	1.00e-03	4.16e-04	1.81e-04
$r_\varepsilon^{N,K}$	1.00	1.26	1.31	1.32	1.27	1.20	

Table 1: Errors in the computed solution to (24) on a Shishkin mesh with $N = K$ subintervals on each axis

of convergence calculated as described above. We note that for small ε, the error is independent of this parameter.

However we also observe that the rates of convergence in Table 1 are not entirely in agreement with the theory: this is because the rate of convergence in space is greater than that in time. So, as N increases, the time error begins to dominate and the rate of convergence is reduced.

In order to verify that the observable the rate of convergence with respect to N is almost second order, as proved in §5.1, we proceed as follows. For a given N we choose K sufficiently large that the spatial error dominates. For example (24) this can be done by taking K to be four times the smallest integer greater than $(N/\ln N)^2$. One may observe that the method converges at a rate that is indeed almost second-order in N. The results are shown in Table 2.

ε^2	$N=2^4$	$N=2^5$	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$	$N=2^{10}$
1	1.36e-03	5.34e-04	1.92e-04	6.49e-05	2.11e-05	6.64e-06	2.04e-06
$r_\varepsilon^{N,K}$	1.35	1.47	1.57	1.62	1.67	1.70	
1e-02	7.13e-03	2.03e-03	5.54e-04	1.50e-04	4.09e-05	1.12e-05	3.09e-06
$r_\varepsilon^{N,K}$	1.81	1.87	1.88	1.88	1.87	1.86	
1e-04	2.34e-02	1.17e-02	4.62e-03	1.65e-03	5.60e-04	1.80e-04	5.60e-05
$r_\varepsilon^{N,K}$	1.01	1.34	1.48	1.56	1.64	1.68	
1e-06	2.35e-02	1.17e-02	4.62e-03	1.65e-03	5.60e-04	1.80e-04	5.60e-05
$r_\varepsilon^{N,K}$	1.01	1.34	1.48	1.56	1.64	1.68	
1e-08	2.35e-02	1.17e-02	4.62e-03	1.65e-03	5.60e-04	1.80e-04	5.60e-05
$r_\varepsilon^{N,K}$	1.01	1.34	1.48	1.56	1.64	1.68	
1e-10	2.35e-02	1.17e-02	4.62e-03	1.65e-03	5.60e-04	1.80e-04	5.60e-05
$r_\varepsilon^{N,K}$	1.01	1.34	1.48	1.56	1.64	1.68	

Table 2: Errors in the computed solution to (24) on a Shishkin mesh with $K = 4\lceil (N/\ln N)^2\rceil$ subintervals on the t-axis

Finally we verify that there is 1st order convergence in time by fixing $N = 16K$ so that so that the temporal error dominates. The results are shown in Table 3 and again verify the theory (22).

6.2 A Bakhvalov mesh

In 4 we present the numerical results for Example (24) where ω_x^N is a Bakhvalov mesh constructed as described in described in §5.2. We take $\kappa = 1$, $\sigma = 2.5$ and $\varrho = 1$. As in the previous sections we take $K = N$ uniform time steps. As predicted in (23) the errors are independent of ε. It is also notable that, compared with Table 1, the Bakhvalov mesh produces a more accurate result.

ε^2	$K=2^4$	$K=2^5$	$K=2^6$	$K=2^7$	$K=2^8$	$N=2^9$	$N=2^{10}$
1	7.98e-03	4.59e-03	2.50e-03	1.31e-03	6.69e-04	3.38e-04	1.70e-04
$r_\varepsilon^{N,K}$	0.80	0.88	0.93	0.97	0.98	0.99	
1e-02	9.01e-03	4.61e-03	2.33e-03	1.17e-03	5.89e-04	2.95e-04	1.47e-04
$r_\varepsilon^{N,K}$	0.97	0.98	0.99	1.00	1.00	1.00	
1e-04	9.23e-03	4.67e-03	2.34e-03	1.17e-03	5.87e-04	2.93e-04	1.47e-04
$r_\varepsilon^{N,K}$	0.98	0.99	1.00	1.00	1.00	1.00	
1e-06	9.23e-03	4.66e-03	2.34e-03	1.17e-03	5.86e-04	2.93e-04	1.47e-04
$r_\varepsilon^{N,K}$	0.98	0.99	1.00	1.00	1.00	1.00	
1e-08	9.22e-03	4.66e-03	2.34e-03	1.17e-03	5.86e-04	2.93e-04	1.47e-04
$r_\varepsilon^{N,K}$	0.98	0.99	1.00	1.00	1.00	1.00	
1e-10	9.22e-03	4.66e-03	2.34e-03	1.17e-03	5.86e-04	2.93e-04	1.47e-04
$r_\varepsilon^{N,K}$	0.98	0.99	1.00	1.00	1.00	1.00	

Table 3: Errors in the computed solution to (24) on a Shishkin mesh with $N = 8K$ subintervals on the x-axis

ε^2	$N=2^4$	$N=2^5$	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$	$N=2^{10}$
	16	32	64	128	256	512	1024
1	8.08e-03	4.61e-03	2.51e-03	1.31e-03	6.69e-04	3.38e-04	1.70e-04
$r_\varepsilon^{N,K}$	0.81	0.88	0.94	0.97	0.98	0.99	
1e-02	9.21e-03	4.67e-03	2.35e-03	1.18e-03	5.90e-04	2.95e-04	1.48e-04
$r_\varepsilon^{N,K}$	0.98	0.99	1.00	1.00	1.00	1.00	
1e-04	9.37e-03	4.70e-03	2.35e-03	1.17e-03	5.86e-04	2.93e-04	1.47e-04
$r_\varepsilon^{N,K}$	1.00	1.00	1.00	1.00	1.00	1.00	
1e-06	9.46e-03	4.71e-03	2.35e-03	1.17e-03	5.86e-04	2.93e-04	1.46e-04
$r_\varepsilon^{N,K}$	1.01	1.00	1.00	1.00	1.00	1.00	
1e-08	9.47e-03	4.71e-03	2.35e-03	1.17e-03	5.86e-04	2.93e-04	1.46e-04
$r_\varepsilon^{N,K}$	1.01	1.00	1.00	1.00	1.00	1.00	
1e-10	9.47e-03	4.71e-03	2.35e-03	1.17e-03	5.86e-04	2.93e-04	1.46e-04
$r_\varepsilon^{N,K}$	1.01	1.00	1.00	1.00	1.00	1.00	

Table 4: Errors in the computed solution to (24) on a Bakhvalov mesh with $N = K$ subintervals on each axis

As with Table 1 the rates of convergence with respect to N are lower than predicted by the theory. This is because we have set $K = N$, and since the method is only first-order in time, the time error dominates. This is particularly obvious if the results are compared with Table 3.

To show that estimate (24) is sharp with respect to N, it is necessary to take K sufficiently large that the spatial error dominates. Similar to Table 2, it suffices to set $K = 4N^2$. The results are presented in Table 5 below, can confirm that the method is indeed second-order in N.

References

[1] V.B. Andreev. On the uniform convergence of a classical difference scheme on an irregular grid for the one-dimensional singularly perturbed reaction-diffusion equation. *Zh. Vychisl. Mat. Mat. Fiz.*, 44(3):476–492, 2004.

[2] N. S. Bakhvalov. Towards optimization of methods for solving boundary value problems in the presence of boundary layers. *Zh. Vychisl. Mat. i Mat. Fiz.*, 9:841–859, 1969. In Russian.

[3] N. Kopteva. Uniform pointwise convergence of difference schemes for convection-diffusion problems on layer-adapted meshes. *Computing*, 66(2):179–197, 2001.

ε^2	$N=2^4$	$N=2^5$	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$	$N=2^{10}$
1	3.05e-04	7.67e-05	1.92e-05	4.80e-06	1.20e-06	3.00e-07	7.50e-08
$r_\varepsilon^{N,K}$	1.99	2.00	2.00	2.00	2.00	2.00	
1e-02	1.20e-03	3.23e-04	8.58e-05	2.21e-05	5.60e-06	1.41e-06	3.55e-07
$r_\varepsilon^{N,K}$	1.90	1.91	1.96	1.98	1.99	1.99	
1e-04	1.93e-03	5.39e-04	1.43e-04	3.73e-05	9.49e-06	2.40e-06	6.04e-07
$r_\varepsilon^{N,K}$	1.84	1.91	1.94	1.97	1.98	1.99	
1e-06	2.07e-03	5.81e-04	1.55e-04	4.03e-05	1.03e-05	2.60e-06	6.54e-07
$r_\varepsilon^{N,K}$	1.83	1.91	1.94	1.97	1.98	1.99	
1e-08	2.09e-03	5.87e-04	1.56e-04	4.07e-05	1.04e-05	2.63e-06	6.62e-07
$r_\varepsilon^{N,K}$	1.83	1.91	1.94	1.97	1.98	1.99	
1e-10	2.09e-03	5.88e-04	1.56e-04	4.08e-05	1.04e-05	2.63e-06	6.63e-07
$r_\varepsilon^{N,K}$	1.83	1.91	1.94	1.97	1.98	1.99	
1e-12	2.09e-03	5.88e-04	1.56e-04	4.08e-05	1.04e-05	2.63e-06	6.63e-07
$r_\varepsilon^{N,K}$	1.83	1.91	1.94	1.97	1.98	1.99	

Table 5: Errors in the computed solution to (24) on a Bakhvalov mesh with $K = 4N^2$ subintervals on each axis

[4] T. Linß. Sufficient conditions for uniform convergence on layer-adapted meshes for one-dimensional reaction-diffusion problems. *Numer. Algorithms*, 40(1):23–32, 2005.

[5] J. J. H. Miller, E. O'Riordan, and G. I. Shishkin. *Fitted numerical methods for singular perturbation problems. Error estimates in the maximum norm for linear problems in one and two dimensions.* World Scientific, Singapore, 1996.

[6] J. J. H. Miller, E. O'Riordan, G. I. Shishkin, and L. P. Shishkina. Fitted mesh methods for problems with parabolic boundary layers. *Math. Proc. R. Ir. Acad.*, 98A(2):173–190, 1998.

[7] G. I. Shishkin. *Grid Approximation of Singularly Perturbed Elliptic and Parabolic Equations.* Second doctorial thesis, Keldysh Institute, Moscow, 1990. In Russian.