A parameter-robust numerical method for a system of reaction-diffusion equations in two dimensions

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11 November 2006

Abstract

A system of $M(\geq 2)$ coupled singularly perturbed linear reaction-diffusion equations is considered on the unit square. Under certain hypotheses on the coupling, a maximum principle is established for the differential operator. The relationship between compatibility conditions at the corners of the square and the smoothness of the solution on the closed domain is fully described. A decomposition of the solution of the system is constructed. A finite difference method for the solution of the system on a Shishkin mesh is presented and it is proved that the computed solution is second-order accurate (up to a logarithmic factor). Numerical results are given to support this result and to investigate the effect of weaker compatibility assumptions on the data.

1 Introduction

Numerical methods for singularly perturbed linear reaction-diffusion problems have received much attention in recent papers such as [1, 3, 7, 11, 12, 13, 14, 18]. More details of the problems considered in these papers will be given below, but no published paper considers a system of singularly perturbed reaction-diffusion problems posed on a 2-dimensional polygonal domain. Such systems appear, for example, in electro-analytical chemistry when investigating diffusion processes in the presence of chemical reactions.

In the present paper we shall consider the following coupled system of M linear second-order singularly perturbed boundary value problems, posed on the unit square: find $\mathbf{u}(x,y) = \big(u_1(x,y), \ldots, u_M(x,y)\big)^T$ that satisfies

$$L\mathbf{u} : = -\varepsilon^2 \Delta \mathbf{u} + A\mathbf{u} = \mathbf{f} \quad \text{on } \Omega := (0,1) \times (0,1), \tag{1.1a}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega, \tag{1.1b}$$

where Δu is the Laplacian, the singular perturbation parameter ε lies in the interval $(0,1]$, and $A = (a_{ij})$ is an $M \times M$ matrix. The other data of the problem are $\mathbf{f}(x,y) = \big(f_1(x,y), \ldots, f_M(x,y)\big)^T$ and $\mathbf{g}(x,y) = \big(g_1(x,y), \ldots, g_M(x,y)\big)^T$; we assume that $\mathbf{f} \in C(\bar{\Omega})$ and $\mathbf{g} \in C(\partial\Omega)$. Precise hypotheses on the entries of A will be given in Assumption 2.1 to ensure that (1.1) has a unique solution in $C(\bar{\Omega}) \cap C^2(\Omega)$: see Corollary 2.1. Compatibility conditions must also be assumed on the data: see Assumption 3.1.

Near the boundary $\partial\Omega$, the solution $\mathbf{u}$ to (1.1) will typically have layers—narrow regions in which $\mathbf{u}$ changes rapidly—when ε is close to zero. Standard numerical methods do not yield accurate solutions for problems of this type. Furthermore, they may fail to resolve the boundary layers, which is usually the part of $\mathbf{u}$ that

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is of greatest interest. It is our aim to present a numerical method for (1.1) that yields a computed solution whose accuracy is independent of ε and that resolves the layers in $\mathbf{u}$. In the research literature, methods that are accurate independently of ε are variously called *parameter robust*, *ε-uniform* or *uniformly convergent*; see [16, 17] and their references. Our numerical method is a standard finite difference scheme that is applied on a special mesh that is constructed a priori.

We now provide a brief summary of previous work on the numerical solution of linear singularly perturbed reaction-diffusion problems. All the papers cited in this section analyse numerical methods on "Shishkin meshes" and these meshes will be the basis of our numerical method also.

A single reaction-diffusion problem on an interval is considered in [15, Chap. 6]. Systems of two reaction-diffusion equations posed on an interval are analysed in [11, 12, 13, 14].

A single reaction-diffusion problem on the unit square was studied in [3]; it was shown in [1, 2] that the compatibility conditions assumed in the hypotheses of [3] could be relaxed. Other approaches to this problem was taken in [6] and [10].

A coupled system posed on an infinite strip was dealt with in [20].

Time-dependent reaction-diffusion problems appear in a few papers: a system in one space variable was studied in [7] and a single equation in two space dimensions is discussed in [4].

In dealing with problem (1.1) we continue the investigations of the above papers. Nevertheless the combination of a non-smooth domain and a coupled system raise issues that do not arise in any of the above papers; in particular we deal with the new question of compatibility of the data at the corners of the domain for a system of equations. Furthermore, for the case $M = 2$ we can weaken the hypotheses on the coupling matrix A that were used in [11, 12, 13]; see Remark 2.1. In a paper [19] that has only recently been brought to our attention, Shishkin considers the case $M = 2$ of our problem (though he remarks that a similar result is true for the general case) and outlines how a result similar to our Theorem 5.1 can be obtained, but his analysis differs from ours in several respects, many details of arguments are missing from [19] (e.g., the convergence result for the finite difference method is stated without proof), the assumptions made there on the compatibility of the data at the corners of the domain are more restrictive than ours, and no numerical results are given.

The structure of our paper is as follows. In Section 2 conditions are imposed on the entries of A that lead to a simple iterative procedure for solving (1.1), and a maximum principle is proved for (1.1). Section 3 examines the crucial issue of compatibility for (1.1) at the corners of the domain and gives general conditions under which the solution $\mathbf{u}$ achieves a desired amount of smoothness on the closed square $\bar{\Omega}$. Then in Section 4 we present a decomposition of $\mathbf{u}$ that reveals its structure and that will be used in its numerical analysis. Section 5 describes the numerical method and in Theorem 5.1 an error estimate for the computed solution is derived. Finally, numerical results are presented in Section 6 in support of the theoretical analysis.

Notation. Throughout this paper we use C to denote a generic constant and $\mathbf{C}$ a generic constant vector that are independent of ε and the mesh. Note that C and $\mathbf{C}$ can take different values in different places. We occasionally use a subscripted C and $\mathbf{C}$ (e.g., $C_1, \mathbf{C}_1$); these are independent of ε and the mesh and are fixed—they do not vary in value.

Set $\|v\|_\infty = \max_{(x,y)\in\bar{\Omega}} |v(x,y)|$ for all $v \in C(\bar{\Omega})$. For a vector-valued function $\mathbf{v} = (v_i)$ with each $v_i \in C(\bar{\Omega})$, set $\|\mathbf{v}\|_\infty = \max_i\{\|v_i\|_\infty\}$. Hölder spaces and their norms are used in Section 3.

2 Jacobi iteration and a maximum principle

Hypotheses must be placed on the entries in A in order to ensure that (1.1) has a solution.

Assumption 2.1. *Throughout the paper we assume that there are positive constants α and β such that*

$$a_{ii} \geq \alpha > 0 \text{ on } \bar{\Omega} \text{ for } i = 1, \cdots, M, \tag{2.1a}$$

$$a_{ij} < 0 \text{ on } \bar{\Omega} \text{ for } i \neq j, \tag{2.1b}$$

and, setting $\beta_i = \max_{\bar\Omega}\left\{a_{ii}(x,y)^{-1}\sum_{j\neq i}|a_{ij}(x,y)|\right\}$ *for* $i=1,\ldots,M$,

$$\beta := \max_i \beta_i < 1. \tag{2.1c}$$

In the case $M=2$ the assumption (2.1c) can be replaced by the weaker condition $\beta_1\beta_2 < 1$. The assumptions (2.1a) and (2.1c) imply that A is diagonally dominant at each point in $\bar\Omega$, and is therefore nonsingular in $\bar\Omega$. If all three assumptions hold, then A^{-1} has non-negative entries. Our first results (Lemma 2.1, Lemma 2.2 and Corollary 2.1) do not in fact require the off-diagonal sign assumption (2.1b). Furthermore, (2.1b) can be replaced by the weaker assumption $a_{ij}\le 0$ on Ω for $i\neq j$ in most of the remainder of the paper; the strict inequality of (2.1b) is used only in the construction of the boundary layer functions $\mathbf{w}_1,\ldots,\mathbf{w}_4$ in Section 4 and could perhaps be weakened to $a_{ij}\le 0$ by an alternative approach there.

Define the decoupled operators

$$L_i v(x,y) := -\varepsilon\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)v(x,y) + a_{ii}(x,y)v(x,y) \quad \text{ for } i=1,\cdots M.$$

Lemma 2.1. *If* $w\in C(\bar\Omega)\cap C^2(\Omega)$, *with* $L_i w = g$ *on* Ω *and* $w=0$ *on* $\partial\Omega$, *then*

$$\|w\|_\infty \le \|g/a_{ii}\|_\infty. \tag{2.2}$$

Proof. Use the barrier function $\|g/a_{ii}\|_\infty$. Clearly this dominates the boundary condition $w=0$, and for all $(x,y)\in\Omega$ we have

$$L_i\|g/a_{ii}\|_\infty(x,y) = a_{ii}(x,y)\|g/a_{ii}\|_\infty \ge a_{ii}(x,y)|(g/a_{ii})(x,y)| = |g(x,y)| = |L_i w(x,y)|.$$

The result follows. ■

In [14] the discrete linear system of equations generated by the numerical method was solved by constructing a sequence of uncoupled problems whose solutions converged to the desired solution. We show in the next result that the same idea works on the continuous level; thus this idea is useful both in theory and in practice.

Lemma 2.2. *Let* $\mathbf{u}$ *be a solution of (1.1). Define the following sequence of vector-valued functions* $\mathbf{u}^{[k]} = (u_1^{[k]}, u_2^{[k]})$ *for* $k=0,1,2,\ldots$ *as follows: let* $\mathbf{u}^{[0]}$ *be any function in* $C(\bar\Omega)\times C(\bar\Omega)$, *and for* $k=1,2,\ldots,$ *let* $\mathbf{u}^{[k]}$ *satisfy*

$$L_i u_i^{[k]} = f_i - \sum_{j\neq i} a_{ij}u_j^{[k-1]} \ \text{ on } \Omega, \qquad u_i^{[k]}(x,y) = g_i(x,y) \ \text{ on } \partial\Omega, \tag{2.3}$$

Then $\lim_{k\to\infty}\mathbf{u}^{[k]} = \mathbf{u}$.

Proof. Set $\mathbf{n}^{[k]} = \mathbf{u}-\mathbf{u}^{[k]}$ for $k=0,1,\ldots$. For $k\ge 1$, we have

$$L_i n_i^{[k]} = -\sum_{j\neq i} a_{ij}n_j^{[k-1]} \text{ on } \Omega, \qquad n_i^{[k]}(x,y) = 0 \text{ on } \partial\Omega \ \text{ for } i=1,\ldots,M.$$

Lemma (2.1) yields

$$\|n_i^{[k]}\|_\infty \le \sum_{j\neq i}\|a_{ij}n_j^{[k-1]}/a_{ii}\|_\infty \le \beta_i\|\mathbf{n}^{[k-1]}\|_\infty.$$

Hence $\|\mathbf{n}^{[k]}\|_\infty \le \beta\|\mathbf{n}^{[k-1]}\|_\infty \le \beta^n\|\mathbf{n}^{[0]}\|_\infty$, so $\|\mathbf{n}^{[k]}\|_\infty \to 0$. ■

Corollary 2.1. *The system* (1.1) *has a solution and this solution is unique.*

Proof. Lemma 2.2 implies that (1.1) has at most one solution, because the sequence $\mathbf{u}^{[k]}$ has been proved to converge to any specified solution of (1.1). In particular if all the data of (1.1) are homogeneous ($\mathbf{f}=\mathbf{g}=\mathbf{0}$) then the only solution is $\mathbf{u}=\mathbf{0}$. Now the standard theory for solutions of systems of linear boundary-value problems [9, §7.5] implies the desired result. ■

To prove convergence of the numerical method that will be constructed in Section 5 we shall use maximum principles, and it is here that assumption (2.1b) enters the game.

Lemma 2.3 (Maximum Principle)**.** *Let* $\mathbf{v} \in C(\bar{\Omega}) \cap C^2(\Omega)$ *with* $L\mathbf{v} \geq \mathbf{0}$ *on* Ω *and* $\mathbf{v} \geq \mathbf{0}$ *on* $\partial\Omega$*. Then* $\mathbf{v} \geq \mathbf{0}$ *on* $\bar{\Omega}$*.*

Proof. Construct the sequence $\mathbf{u}^{[1]}, \mathbf{u}^{[2]}, \ldots$ of solutions to the variant of the uncoupled problems (2.3) where $\mathbf{u}^{[0]} = \mathbf{0}$, $\mathbf{f}$ is replaced by $L\mathbf{v}$ and $\mathbf{g}$ by $\mathbf{v}|_{\partial\Omega}$. The proof of Lemma 2.2 shows that $\lim_{k\to\infty} \mathbf{u}^{[k]} = \mathbf{v}$. When $k = 1$,

$$L_1 u_i^{[1]} = (Lv)_i \geq 0 \text{ in } \Omega \quad \text{and} \quad u_i^{[1]} = v_i \geq 0 \text{ on } \partial\Omega.$$

The maximum principle satisfied by L_i then implies that $\mathbf{u}^{[1]} \geq 0$. Since the off-diagonal entries of A are non-positive,

$$L_i u_i^{[2]} = (L\mathbf{v})_i - \sum_{j\neq i} a_{ij} u_j^{[1]} \geq 0, \quad \text{and } u_i^{[2]} = v_i \geq 0 \text{ on } \Gamma.$$

Again using the maximum principle for the operator L_i, $u_i^{[2]} \geq 0$, so $\mathbf{u}^{[2]} \geq 0$. Repeating the above argument, one sees that $\mathbf{u}^{[k]} \geq \mathbf{0}$ for all k. Consequently $\mathbf{v} = \lim_{k\to\infty} \mathbf{u}^{[k]} \geq \mathbf{0}$. ■

We may think of the conditions (2.1a) and (2.1c) as ensuring the stability of L, while if (2.1b) is also assumed, then in addition L satisfies a maximum principle.

Remark 2.1. *Assumption 2.1 has been used in other papers on linear singularly perturbed reaction-diffusion problems, e.g, [12, 13, 14], all of which are concerned with systems of just two equations. In that case, one may however replace* (2.1c) *by the weaker assumption*

$$\beta_1\beta_2 < 1. \tag{2.4}$$

It can be shown that in the case $M = 2$*, the assumptions* (2.1a), (2.1b) *and* (2.4) *are sufficient to imply Lemma 2.1, Lemma 2.2, Corollary 2.1 and Lemma 2.3.*

Example 2.1. *One might wonder if the only property needed from* A *is that it be inverse monotone. This is not the case. For consider the example*

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $A^{-1} = A$ *so* $A^{-1} \geq 0$*. Now* (1.1a) *becomes*

$$-\varepsilon\Delta u_1 + u_2 = f_1, \quad -\varepsilon\Delta u_2 + u_1 = f_2.$$

Set $v_1 = u_1 + u_2,\ v_2 = u_1 - u_2$*. Adding and subtracting the two differential equations gives the uncoupled system*

$$-\varepsilon\Delta v_1 + v_1 = f_1 + f_2, \quad -\varepsilon\Delta v_2 - v_2 = f_1 - f_2.$$

Here v_1 *is the type of solution that we expect, but* v_2 *will in general be highly oscillatory and consequently the decomposition of* $\mathbf{u}$ *into a sum of smooth components and layers in Section 4 will fail.*

3 Compatibility conditions and regularity of the solution

The solution to (1.1) may in general have singularities near the four corners of $\Omega = (0,1) \times (0,1)$. The purpose of this section is to give a brief discussion of these "corner singularities" and conditions on $\mathbf{f}$ that exclude them from the solution.

It is convenient to work in a framework of Hölder spaces. Let k be a non-negative integer and let $\sigma \in (0,1)$. We say that $v \in C^{k,\sigma}(\bar{\Omega})$ if all the derivatives of v up to order k are continuous in $\bar{\Omega}$ and if each of the derivatives of order k is Hölder continuous with exponent σ. We say that $\mathbf{g} \in C^{k,\sigma}(\partial\Omega)$ if $\mathbf{g}$ is continuous at the four vertices of $\bar{\Omega}$ and if $\mathbf{g} \in C^{k,\sigma}$ on each of the closed line segments that constitute the boundary $\bar{\Omega}$. Note that these are Banach spaces with appropriate norms.

The general result is as follows; the discussion in the rest of this section sketches a proof.

Theorem 3.1. *Let $\nu \geq 0$ be an integer. There are $4(\nu+1)M$ bounded linear functionals on $C^{2\nu,\sigma}(\bar{\Omega}) \times C^{2\nu+2,\sigma}(\partial\Omega)$, denoted by $\Lambda_{A,\mu,i,\ell}$ for $\mu = 0,\cdots,\nu$, $i = 1,\cdots,M$, $\ell = 1,\cdots,4$, such that if $\mathbf{f} \in C^{2\nu,\sigma}(\bar{\Omega})$, $\mathbf{g} \in C^{2\nu+2}(\partial\Omega)$ and $\mathbf{u}$ is a solution of (1.1), then $\mathbf{u} \in C^{2\nu+2,\sigma}(\bar{\Omega})$ if and only if*

$$\Lambda_{A,\mu,i,\ell}(\mathbf{f},\mathbf{g}) = 0 \text{ for } \mu = 0,\cdots,\nu,\ i = 1,\cdots,M,\ \ell = 1,\cdots,4. \tag{3.1}$$

If (3.1) holds, then

$$\|\mathbf{u}\|_{C^{2\nu+2,\sigma}} \leq C\varepsilon^{-(2\nu+2)}\left(\|\mathbf{f}\|_{C^{2\nu,\sigma}(\bar{\Omega})} + \|\mathbf{g}\|_{C^{2\nu+2,\sigma}(\partial\Omega)}\right). \tag{3.2}$$

If (3.1) holds and furthermore $\mathbf{f} \in C^{2\nu+1,\sigma}(\bar{\Omega})$ and $\mathbf{g} \in C^{2\nu+3,\sigma}(\partial\Omega)$, then $\mathbf{u} \in C^{2\nu+3,\sigma}(\bar{\Omega})$ with an analogous inequality.

As we shall see, the corner singularities arise from incompatibilities between the boundary data and the differential equations. In the case that A is a constant matrix, it turns out that these linear functionals are "local" in that they depend only on linear combinations of derivatives of the components of $\mathbf{f}$ and $\mathbf{g}$ evaluated at the four vertices of Ω. As a consequence, if $\mathbf{g} = \mathbf{0}$ and $\mathbf{f}$ and its derivatives up to order 2ν vanish at the four vertices of Ω then $\mathbf{u} \in C^{2\nu+2,\sigma}(\bar{\Omega})$. If A is not a constant matrix, then the linear functionals are not in general local, but the first few linear functionals are still local, and this enables one to deduce low-order regularity of $\mathbf{u}$ from the vanishing of $\mathbf{f}$ and its derivatives (in the case $\mathbf{g} = \mathbf{0}$) at the vertices of Ω. The following discussion sketches a proof of these assertions, and of Theorem 3.1.

We start the discussion with the scalar problem

$$\Delta u = f \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega. \tag{3.3}$$

If $f \in C^{0,\sigma}(\bar{\Omega})$ there is an incompatibility between the boundary conditions and the differential equation at vertex ℓ of Ω unless $\bar{\Lambda}_{0,\ell}(f) := f = 0$ at vertex ℓ. Similarly, if $f \in C^{2,\sigma}(\bar{\Omega})$ there is an incompatibility between the boundary conditions and the differential equation at vertex ℓ unless $\bar{\Lambda}_{1,\ell}(f) := f_{xx} - f_{yy} = 0$ at vertex ℓ. Pursuing these calculations, one arrives at the linear functionals

$$\bar{\Lambda}_{\mu,\ell}(f) := \sum_{j=0}^{\mu}(-1)^j D_x^{2j} D_y^{2(\mu-j)} f \quad \text{at vertex } \ell. \tag{3.4}$$

Here D_x and D_y denote partial derivatives with respect to the x and y variables.

In [8] the following result is shown.

Theorem 3.2. *If $f \in C^{2\nu,\sigma}(\bar{\Omega})$ and u is a solution of (3.3), then $u \in C^{2\nu+2,\sigma}(\bar{\Omega})$ if and only if $\bar{\Lambda}_{\mu,\ell}(f) = 0$ for $\mu = 0,\cdots,\nu$, $\ell = 1,\cdots,4$. In this event, $\|\mathbf{u}\|_{C^{2\nu+2,\sigma}(\bar{\Omega})} \leq C\|f\|_{C^{2\nu,\sigma}(\bar{\Omega})}$. If in addition $f \in C^{2\nu+1,\sigma}(\bar{\Omega})$, then $u \in C^{2\nu+3,\sigma}(\bar{\Omega})$ and an analogous inequality holds.*

For example, if $f = 0$ at the four vertices of Ω, then $u \in C^{2,\sigma}(\bar{\Omega})$, and if $f = f_{yy} - f_{xx} = 0$ at the four vertices of Ω, then $u \in C^{4,\sigma}(\bar{\Omega})$. There is also an algebraic consequence of the vanishing of the linear functionals that is important in extending the analysis of (3.3) to (1.1). Namely,

$$\begin{gathered}\text{if } \bar{\Lambda}_{\mu,\ell}(f) = 0 \text{ for } \mu = 0,\cdots,\nu,\ \ell = 1,\cdots,4, \text{ then the mixed derivatives } D_x^{2j} D_y^{2k} u \\ \text{with } j + k = \nu + 1, \text{ evaluated at vertex } \ell, \text{ can be expressed in terms of} \\ \text{derivatives of } f \text{ of the form } D_x^{2j} D_y^{2k} f \text{ with } j + k = \nu, \text{ evaluated at vertex } \ell.\end{gathered} \tag{3.5}$$

For example, differentiating the differential equation twice and using some algebra gives $u_{xxyy} = f_{xx} = f_{yy}$ at vertex ℓ if $\bar{\Lambda}_{1,\ell} f = f_{yy} - f_{xx} = 0$ at vertex ℓ.

We now derive compatibility conditions for the problem (1.1) in the case $\mathbf{g} = 0$. The i-th component u_i of the solution satisfies the boundary value problem

$$\Delta u_i = F_i = \varepsilon^{-2}(A\mathbf{u})_i - \varepsilon^{-2} f_i \text{ in } \Omega,\ u_i = 0 \text{ on } \partial\Omega. \tag{3.6}$$

Suppose $\mathbf{f}$ belongs to $C^{0,\sigma}(\bar{\Omega})$, and let $\mathbf{u}$ be a solution of (1.1). From Theorem 3.2 applied to the problem (3.6) we have $u_i \in C^{2,\sigma}(\bar{\Omega})$ if and only if $F_i = -f_i = 0$ at the 4 vertices, so we define

$$\Lambda_{A,0,i,\ell}(\mathbf{f},0) = f_i \quad \text{at vertex } \ell. \tag{3.7}$$

Next, suppose $\Lambda_{A,0,i,\ell}(\mathbf{f},0) = 0$ for $i = 1, \cdots, M$ and $\ell = 1, \cdots, 4$, so from Theorem 3.2 one has $\mathbf{u} \in C^{2,\sigma}(\bar{\Omega})$ and $\|D^2 u_i\|_\infty \le C\varepsilon^{-2}$. (We use $D^r\phi$ to indicate any rth-order derivative of any function ϕ.) Since $\|u_i\|_\infty \le C$, an interpolation argument gives $\|Du_i\|_\infty \le C\varepsilon^{-1}$. Again invoking Theorem 3.2, one sees that $u_i \in C^{4,\sigma}(\bar{\Omega})$ if and only if

$$\partial_y^2 F_i - \partial_x^2 F_i = \varepsilon^{-2}\partial_y^2(A\mathbf{u})_i - \varepsilon^{-2}\partial_x^2(A\mathbf{u})_i - \varepsilon^{-2}\partial_y^2 f_i + \varepsilon^{-2}\partial_x^2 f_i = 0 \quad \text{at the four vertices.}$$

The boundary conditions yield $\partial_y^2(A\mathbf{u})_i - \partial_x^2(A\mathbf{u})_i = 0$ at the vertices, so we define

$$\Lambda_{A,1,i,\ell}(\mathbf{f},0) = \partial_y^2 f_i - \partial_x^2 f_i \quad \text{at vertex } \ell. \tag{3.8}$$

The inequality of Theorem 3.2 gives $\|D^4 u_i\|_{C^{4,\sigma}(\bar{\Omega})} \le C\varepsilon^{-2}(\|f_i\|_{C^{2,\sigma}(\bar{\Omega})} + \|\mathbf{u}\|_{C^{2,\sigma}(\bar{\Omega})}) \le C\varepsilon^{-4}$. An interpolation argument then gives $\|D^3\mathbf{u}\|_\infty \le C\varepsilon^{-3}$. Summarizing, we have shown the following particular case of Theorem 3.1:

$$\begin{gathered}\text{if } \mathbf{f} \in C^{2,\sigma}(\bar{\Omega}) \text{ and if } \mathbf{f} = \partial_y^2\mathbf{f} - \partial_x^2\mathbf{f} = \mathbf{0} \text{ at each of the four corners of } \Omega,\\ \text{then } \mathbf{u} \in C^{4,\sigma}(\bar{\Omega}) \text{ and } \|D^m\mathbf{u}\|_\infty \le C\varepsilon^{-m} \text{ for } m = 0, \cdots, 4.\end{gathered} \tag{3.9}$$

Continuing in this way, an inductive argument gives formulas for the higher-order linear functionals and inequalities for the solution. If A is constant one finds, using (3.5), that the linear functionals depend only on derivatives of f at the vertices. If A is not constant, the higher-order linear functionals involve derivatives of $\mathbf{u}$ other than those specified in (3.5). These derivatives cannot be evaluated in terms of derivatives of $\mathbf{f}$ at the origin, and the linear functionals no longer have a local character.

To derive compatibility conditions for the problem (1.1) in the case $\mathbf{g} \neq 0$, we perform a reduction to the case $\mathbf{g} = 0$ using an extension operator $E_k : C^{k,\sigma}(\partial\Omega) \to C^{k,\sigma}(\bar{\Omega})$. Such an extension operator is easily constructed, and is a bounded linear map between the two Banach spaces. Let $\mathbf{f} \in C^{2\nu,\sigma}(\bar{\Omega})$, $\mathbf{g} \in C^{2\nu+2}(\partial\Omega)$, and let $\mathbf{u}$ solve (1.1). Then $\mathbf{u}^* = \mathbf{u} - E_{2\nu+2}\mathbf{g}$ vanishes on $\partial\Omega$ and satisfies $L\mathbf{u}^* = \mathbf{f}^* := \mathbf{f} - LE_{2\nu+2}\mathbf{g}$. Also $\mathbf{f}^* \in C^{2\nu,\sigma}(\bar{\Omega})$. We define

$$\Lambda_{A,\mu,i,\ell}(\mathbf{f},\mathbf{g}) = \Lambda_{A,\mu,i,\ell}(\mathbf{f}^*,0).$$

With this definition one sees that Theorem 3.1 holds in the case of non-zero $\mathbf{g}$. One can show that $\Lambda_{A,\mu,i,\ell}(\mathbf{f},\mathbf{g})$ is independent of the choice of extension operator $E_{2\nu+2}$.

In [3] (where $M = 1$) the authors assume that at each corner of Ω one has continuity of the boundary data and also $\bar{\Lambda}_{\mu,\ell}(f,g) = 0$ for $\mu = 0, 1$, but in [2] it is shown that almost the same order of convergence can be established for the numerical method without any hypothesis on the $\bar{\Lambda}_{\mu,\ell}(f,g)$. For our coupled system (1.1) we shall assume that $\Lambda_{A,\mu,i,\ell}(\mathbf{f},\mathbf{g}) = \mathbf{0}$ for $\mu = 0$.

Assumption 3.1. *Assume henceforth that $\mathbf{f} \in C^{2,\sigma}(\bar{\Omega})$, $\mathbf{g} \in C^{4,\sigma}(\partial\Omega)$, and that the data of* (1.1) *is such that $\Lambda_{A,0,i,\ell}(\mathbf{f},\mathbf{g}) = \mathbf{0}$ at each corner of the domain Ω, i.e. that $L\mathbf{g} = \mathbf{f}$ at each corner, where $\mathbf{g}$ here denotes a function that is an extension of the boundary data to $C^{2,\sigma}(\bar{\Omega})$.*

By Theorem 3.1, this assumption implies that $\mathbf{u} \in C^3(\bar{\Omega})$ and

$$\|D^m\mathbf{u}\|_\infty \le C\varepsilon^{-m} \text{ for } m = 0, 1, 2, 3. \tag{3.10}$$

In [21] Volkov (see also [2]) observed that for the problem (3.3), while the compatibility condition $\bar{\Lambda}_{1,\ell}(f,g) = 0$ at each corner (with sufficient smoothness of the data f, g) is in general insufficient to yield $u \in C^4(\bar{\Omega})$, an inspection of the singularities of the solution (cf. [8]) shows that nevertheless one has $D_x^{2j}D_y^{4-2j}u \in C^4(\bar{\Omega})$ for $j = 0, 1, 2$. As the compatibility analysis of the system (1.1) carried out in this section proceeds equation by equation, the same observation applies here: Assumption 3.1 implies that

$$D_x^{2j}D_y^{4-2j}\mathbf{u} \in C^4(\bar{\Omega}) \text{ with } \|D_x^{2j}D_y^{4-2j}\mathbf{u}\|_\infty \le C\varepsilon^{-4} \text{ for } j = 0, 1, 2. \tag{3.11}$$

4 A priori bounds on the solution

The analysis in this section is based on the analysis of a single reaction-diffusion equation given in [3, Section 2], but some simplifications have been achieved.

Notation. Let $\mathbf{1}$ denote the column vector $(1,\dots,1)^T$ in $\mathbb{R}^M$. Given two vector functions $\mathbf{v}(x,y)=(v_1(x,y),\dots,v_M(x,y))^T$ and $\mathbf{w}(x,y)=(w_1(x,y),\dots,w_M(x,y))^T$, by $\mathbf{v}(x,y)\le\mathbf{w}(x,y)$ we mean that $v_i(x,y)\le w_i(x,y)$ for $i=1,\cdots,M$. If $\|v_i\|_{L_\infty(\Omega')}$ is defined for $i=1,\cdots,M$ on some domain Ω', then we set $\|\mathbf{v}\|_{\infty,\Omega'}=\max_i\{\|v_1\|_{L_\infty(\Omega')}\}$.

Before embarking on the details of a decomposition of u, we prove a lemma for certain barrier functions that will be used later.

Lemma 4.1. (barrier functions)

(i) There exists a constant vector function $\mathbf{C}_1>\mathbf{0}$ such that $(L\mathbf{C}_1)(x,y)\ge\mathbf{1}$ on $\bar\Omega$.

(ii) One can choose a constant $C_2>0$ and a vector function $\mathbf{d}(x,y)$ such that $\mathbf{0}<\mathbf{d}(x,y)\le e^{-C_2y/\varepsilon}\mathbf{C}$ for some $\mathbf{C}>\mathbf{0}$ and $L\mathbf{d}(x,y)\ge e^{-C_2y/\varepsilon}\mathbf{1}$ on Ω.

(iii) One can choose a vector function $\mathbf{r}(x,y)$ such that $\mathbf{0}<\mathbf{r}(x,y)\le e^{-C_2x/\varepsilon}e^{-C_2y/\varepsilon}\mathbf{C}$ for some $\mathbf{C}>\mathbf{0}$ and $L\mathbf{r}(x,y)\ge e^{-C_2x/\varepsilon}e^{-C_2y/\varepsilon}\mathbf{1}$ on Ω.

Proof. First consider (i). Set $\alpha=\min_i\min_{(x,y)\in\bar\Omega}a_{ii}(x,y)$. Then $\alpha>0$ by (2.1). Let $C_1=[(1-\beta)\alpha]^{-1}$, so $C_1>0$. Set $\mathbf{C}_1=(C_1,\dots,C_1)$. For each i,

$$(A\mathbf{C}_1)_i(x,y)=C_1\sum_{j=1}^{M}a_{ij}(x,y)=C_1a_{ii}(x,y)\left[1-\frac{\sum_{j\ne i}|a_{ij}(x,y)|}{a_{ii}(x,y)}\right]\ge C_1a_{ii}(x,y)[1-\beta_i]\ge C_1\alpha[1-\beta]=1.$$

Hence $(L\mathbf{C}_1)(x,y)=(A\mathbf{C}_1)(x,y)\ge\mathbf{1}$ on $\bar\Omega$.

For (ii) and (iii), choose $C_2>0$ such that $2C_2^2C_1<1$. Then

$$L(e^{-C_2y/\varepsilon}\mathbf{C}_1)=e^{-C_2y/\varepsilon}(-C_2^2\mathbf{C}_1+A\mathbf{C}_1)\ge e^{-C_2y/\varepsilon}(\mathbf{1}-C_2^2\mathbf{C}_1)$$

which has positive components. Thus, setting $\mathbf{d}=Ce^{-C_2y/\varepsilon}\mathbf{C}_1$ for a suitable constant C, we get (ii).

The proof of (iii) is similar. ∎

Applying this lemma to (1.1), we get

$$\|u\|_{\infty,\bar\Omega}\le C. \tag{4.1}$$

Let Ω^* be the disc in the (x,y)-plane with centre (0,0) and radius 2, so Ω^* is a domain with smooth boundary that contains Ω. Define smooth extensions a_{ij}^* and $\mathbf{f}^*$ of the a_{ij} and $\mathbf{f}$ to Ω^*. This can be done in such a way that A^*, like A, is invertible. Let $\mathbf{v}^*=\mathbf{v}_0^*+\varepsilon^2\mathbf{v}_1^*$, where $\mathbf{v}_0^*$ is the solution of the extended reduced problem $A^*\mathbf{v}_0^*=\mathbf{f}^*$ and $\mathbf{v}_1^*$ is the solution of the boundary value problem

$$L^*\mathbf{v}_1^*=\Delta\mathbf{v}_0^*\text{ on }\Omega^*,\qquad \mathbf{v}_1^*=\mathbf{0}\text{ on }\partial\Omega^*. \tag{4.2}$$

Now $\mathbf{v}_0^*\in C^{4,\sigma}(\Omega^*)$ so $\Delta\mathbf{v}_0^*\in C^{2,\sigma}(\Omega^*)$. Hence $\mathbf{v}_1^*\in C^{4,\sigma}(\Omega^*)$. Observe that by our construction, $L^*\mathbf{v}^*=\mathbf{f}^*$ on Ω^*. Clearly $\|D_x^mD_y^n\mathbf{v}_0^*\|_{\infty,\Omega^*}\le C$ for $0\le m+n\le4$. Using an extension of Lemma 4.1 we deduce that $\|\mathbf{v}_1\|_{\infty,\Omega^*}\le C$. Hence, applying the Schauder estimate of [9, Section 7.5] to the problem (4.2) written in the stretched variables $(x/\varepsilon,y/\varepsilon)$, we get $\|D_x^mD_y^n\mathbf{v}_1^*\|_{\infty,\Omega^*}\le C\varepsilon^{-m-n}$ for $0\le m+n\le4$. As $\mathbf{v}^*=\mathbf{v}_0^*+\varepsilon^2\mathbf{v}_1^*$, it follows that

$$\|D_x^mD_y^n\mathbf{v}^*\|_{\infty,\Omega^*}\le C[1+\varepsilon^{2-m-n}]\text{ for }0\le m+n\le4. \tag{4.3}$$

Letting $\mathbf{v}$ denote the restriction of $\mathbf{v}^*$ to Ω, this gives

$$\|D_x^mD_y^n\mathbf{v}\|_{\infty,\bar\Omega}\le C[1+\varepsilon^{2-m-n}]\text{ for }0\le m+n\le4. \tag{4.4}$$

Let $\Gamma_1 = \{(x,0) : 0 \le x \le 1\}$ denote the bottom edge of $\bar{\Omega}$. We now define a boundary function w_1 associated with Γ_1. Set $\tilde{\Omega} := (-1,2) \times (0,1)$; this rectangular domain contains Ω. Extend the a_{ij} smoothly to functions $\tilde{a}_{ij}$ on $\tilde{\Omega}$ while requiring the signs of the a_{ij} to be preserved and with $(\tilde{a}_{ij})_x = 0$ on the sides $x = -1$ and $x = 2$ of $\tilde{\Omega}$. Write $\tilde{L}$ for the extension of L to $\tilde{\Omega}$. Define $\mathbf{w}$ to be the solution of the boundary value problem

$$\tilde{L}\mathbf{w} = \mathbf{0} \text{ on } \tilde{\Omega}, \tag{4.5a}$$

$$\mathbf{w} = \mathbf{u} - \mathbf{v} \text{ on } \Gamma_1, \tag{4.5b}$$

$$\mathbf{w} = \mathbf{0} \text{ on } \partial\tilde{\Omega} \setminus \{(x,0) : -1 \le x \le 2\}, \tag{4.5c}$$

and on $\{(x,0) : -1 \le x \le 2\} \setminus \Gamma_1$ the value of $\mathbf{w}$ is chosen as a smooth extension of the above boundary conditions; furthermore, all these extensions should be chosen so as to ensure sufficient compatibility at the corners of $\tilde{\Omega}$ to yield $\mathbf{w} \in C^{4,\sigma}(\bar{\tilde{\Omega}})$. Once again using stretched variables and Schauder estimates, we get

$$\|D_x^m D_y^n \mathbf{w}\|_{\infty,\tilde{\Omega}} \le C\varepsilon^{-m-n} \text{ for } 0 \le m+n \le 4. \tag{4.6}$$

Furthermore, $\mathbf{w}$ decays exponentially away from $x = 0$: (4.1) and (4.4) bound $|\mathbf{w}_1|$ on the side $x = 0$ of $\tilde{\Omega}$, then the maximum principle of Lemma 2.3 and a barrier function from Lemma 4.1(ii) yield

$$|\mathbf{w}(x,y)| \le Ce^{-C_2 y/\varepsilon}\mathbf{1} \qquad \text{on } \bar{\Omega}. \tag{4.7}$$

We now derive estimates sharper than (4.6) for $D_x^3\mathbf{w}$ and $D_x^4\mathbf{w}$.

Observe that (4.5a) and (4.5c) imply that

$$D_x^2\mathbf{w}(-1,y) = D_x^2\mathbf{w}(2,y) = \mathbf{0} \;\text{ for }\; 0 \le y \le 1. \tag{4.8}$$

From (4.5b), (4.5c), (1.1b) and (4.3) we have $|D_x^2\mathbf{w}(x,0)| \le C$ and $|D_x^2\mathbf{w}(x,1)| \le C$ for $-1 \le x \le 2$. Hence, using a maximum principle on $\tilde{\Omega}$ with a constant barrier function from Lemma 4.1(i), we get

$$\|D_x^2\mathbf{w}\|_{\infty,\tilde{\Omega}} \le C. \tag{4.9}$$

Applying D_y^2 to (4.5a) and invoking (4.5c) yields $D_x^2 D_y^2\mathbf{w}(-1,y) = D_x^2 D_y^2\mathbf{w}(2,y) = \mathbf{0}$ for $0 \le y \le 1$. Now apply D_x^2 to (4.5a) then invoke (4.8), while recalling that $(\tilde{a}_{ij})_x$ vanishes on the left and right-hand sides of $\tilde{\Omega}$; this yields

$$D_x^4\mathbf{w}(-1,y) = D_x^4\mathbf{w}(2,y) = \mathbf{0} \text{ for } 0 \le y \le 1.$$

From (4.5b), (4.5c), (1.1b) and (4.3) we have $|D_x^4\mathbf{w}(x,0)| \le C\varepsilon^{-2}$ and $|D_x^4\mathbf{w}(x,1)| \le C$ for $-1 \le x \le 2$. Hence, using a maximum principle on $\tilde{\Omega}$ with a constant barrier function from Lemma 4.1(i), we get

$$\|D_x^4\mathbf{w}\|_{\infty,\tilde{\Omega}} \le C\varepsilon^{-2}. \tag{4.10}$$

Next, we interpolate between (4.9) and (4.10) to bound $D_x^3\mathbf{w}$. Fix $(x^*,y^*) \in \tilde{\Omega}$. Choose an interval $[x^-,x^+] \subset [-2,1]$ such that $x^* \in [x^-,x^+]$ and $x^+ - x^- = \varepsilon$. Write $\mathbf{w} = (w_1,w_2)$. By the mean value theorem there exists $\hat{x} \in [x^-,x^+]$ such that

$$D_x^3 w_1(\hat{x},y^*) = \frac{D_x^2 w_1(x^+,y^*) - D_x^2 w_1(x^-,y^*)}{\varepsilon}.$$

Hence, using (4.9), we get $|D_x^3 w_1(\hat{x},y^*)| \le C/\varepsilon$. From this inequality, (4.10) and $x^+ - x^- = \varepsilon$ it follows that

$$|D_x^3 w_1(x^*,y^*)| = \left| \int_{\hat{x}}^{x^*} D_x^4 w_1(s,y^*)\,ds + D_x^3 w_1(\hat{x},y^*) \right| \le C\varepsilon^{-1}.$$

But (x^*,y^*) was an arbitrary point, and a similar argument can be applied to w_2, so we have shown that

$$\|D_x^3\mathbf{w}\|_{\infty,\tilde{\Omega}} \le C\varepsilon^{-1}. \tag{4.11}$$

Now define the boundary layer function $\mathbf{w}_1$ associated with Γ_1 by

$$\begin{aligned} L\mathbf{w}_1 &= 0 \text{ on } \Omega, \\ \mathbf{w}_1 &= \mathbf{u} - \mathbf{v} \text{ on } \Gamma_1, \\ \mathbf{w}_1(x,1) &= \mathbf{0} \text{ on } \{(x,1) : 0 \le x \le 1\}, \\ \mathbf{w}_1(0,y) &= \mathbf{w}(0,y) \text{ and } \mathbf{w}_1(1,y) = \mathbf{w}(1,y) \text{ for } 0 \le y \le 1. \end{aligned}$$

Then $\mathbf{w}_1 = \mathbf{w}$ on $\bar{\Omega}$. Consequently $\mathbf{w}_1 \in C^{4,\sigma}(\bar{\Omega})$ and (4.6), (4.7), (4.10) and (4.11) imply that

$$\|D_x^m D_y^n \mathbf{w}_1\|_{\infty,\bar{\Omega}} \le C\varepsilon^{-m-n} \text{ for } 0 \le m+n \le 4, \tag{4.12a}$$

$$\varepsilon\|D_x^3 \mathbf{w}_1\|_{\infty,\bar{\Omega}} + \varepsilon^2\|D_x^4 \mathbf{w}_1\|_{\infty,\bar{\Omega}} \le C, \tag{4.12b}$$

$$|\mathbf{w}_1(x,y)| \le Ce^{-C_2 y/\varepsilon}\mathbf{1} \qquad \text{on } \bar{\Omega}. \tag{4.12c}$$

Boundary layer functions $\mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ associated with the other three sides of $\bar{\Omega}$ (numbered clockwise) are defined similarly and bounds analogous to (4.12) can be derived.

Define a corner layer $\mathbf{z}_1$ associated with the corner (0,0) of $\bar{\Omega}$ by

$$\begin{aligned} L\mathbf{z}_1 &= \mathbf{0} \text{ on } \Omega, \\ \mathbf{z}_1(0,y) &= -\mathbf{w}_1(0,y) \text{ for } 0 \le y \le 1, \\ \mathbf{z}_1(x,0) &= -\mathbf{w}_2(x,0) \text{ for } 0 \le x \le 1, \\ \mathbf{z}_1 &= \mathbf{0} \text{ on the rest of } \partial\Omega. \end{aligned}$$

To verify compatibility at the corner (0,0), observe that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{u}-\mathbf{v} \in C^{3,\sigma}(\bar{\Omega})$ with $L\mathbf{w}_1 = L\mathbf{w}_2 = L(\mathbf{u}-\mathbf{v}) = \mathbf{0}$, so all three of these functions satisfy $\Lambda_{A,0,i,\ell}(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}) = \mathbf{0}$ at each corner, where $\tilde{\mathbf{f}}, \tilde{\mathbf{g}}$ are the associated data in each case; hence $-\mathbf{z}_1(0,y) = \mathbf{w}_1(0,y)$ is compatible with $\mathbf{w}_1(x,0) = (\mathbf{u}-\mathbf{v})(x,0)$, which is compatible with $(\mathbf{u}-\mathbf{v})(0,y) = \mathbf{w}_2(0,y)$, which is compatible with $\mathbf{w}_2(x,0) = -\mathbf{z}_2(x,0)$, so for the problem defining $\mathbf{z}_1$ the compatibility condition $\Lambda_{A,0,i,\ell}(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}) = \mathbf{0}$ is satisfied at (0,0). Thus $\mathbf{z}_1 \in C^{3,\sigma}(\bar{\Omega})$ by Theorem 3.1. By extending the domain Ω to $(0,2)\times(0,2)$ we can in the usual way appeal to Schauder estimates to get

$$\|D_x^m D_y^n \mathbf{z}_1\|_{\infty,\bar{\Omega}} \le C\varepsilon^{-m-n} \text{ for } 0 \le m+n \le 3. \tag{4.13}$$

The last paragraph of Section 3 yields also

$$D_x^{2j} D_y^{4-2j}\mathbf{z}_1 \in C^4(\bar{\Omega}) \text{ with } \|D_x^{2j} D_y^{4-2j}\mathbf{z}_1\|_\infty \le C\varepsilon^{-4} \text{ for } j = 0,1,2. \tag{4.14}$$

Finally, the bound (4.12c) and an analogous bound for $\mathbf{w}_2$, the maximum principle of Lemma 2.3 and a barrier function from Lemma 4.1(iii) yield the bound

$$|\mathbf{z}_1(x,y)| \le Ce^{-C_2 x/\varepsilon}e^{-C_2 y/\varepsilon}\mathbf{1} \qquad \text{on } \bar{\Omega}. \tag{4.15}$$

Corner layer functions $\mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4$ associated with the other three corners of $\bar{\Omega}$ are defined similarly and bounds analogous to (4.15) and (4.13) can be derived.

To finish, we note that the decomposition of $\mathbf{u}$ that was constructed in this section is

$$\mathbf{u} = \mathbf{v} + \sum_{k=1}^{4}\mathbf{w}_k + \sum_{k=1}^{4}\mathbf{z}_k. \tag{4.16}$$

5 The numerical method and its analysis

We approximate the solution to (1.1) by applying a standard finite difference method on a piecewise uniform "Shishkin" mesh.

To construct the mesh, recall from Assumption 2.1 that we have set

$$\alpha = \min_i \min_{(x,y)\in\bar{\Omega}} a_{ii}(x,y) \qquad \text{and} \qquad \beta = \max_i \max_{\bar{\Omega}} \left\{ a_{ii}(x,y)^{-1} \sum_{j\neq i} |a_{ij}(x,y)| \right\}.$$

Then the *mesh transition point* is

$$\tau_\varepsilon = \min\{1/4,\ 2\sqrt{2}\,(\varepsilon/C_2)\ln N\}, \tag{5.1}$$

where, as given in the proof of Lemma 4.1, we choose C_2 so that

$$C_2 < \sqrt{\frac{\alpha(1-\beta)}{2}}.$$

Remark 5.1. *Recall from Assumption 2.1 that if $M = 2$ we may weaken the assumptions on A to*

$$\beta = \beta_1\beta_2 < 1 \quad \textit{instead of} \quad \beta = \max\{\beta_1, \beta_2\} < 1.$$

In this case, choose C_2 so that

$$C_2 < \sqrt{\frac{\alpha\gamma \min\{\beta_1+\gamma, \beta_2+\gamma\}}{2}},$$

where $\gamma > 0$ is such that $(\beta_1+\gamma)(\beta_2+\gamma) = 1$.

Divide $[0,1]$ into subintervals $[0,\tau_\varepsilon]$, $[\tau_\varepsilon, 1-\tau_\varepsilon]$ and $[1-\tau_\varepsilon, 1]$. A piecewise-uniform mesh $\bar{\omega}_x^N = \{x_i\}_0^N$ is constructed by subdividing $[\tau_\varepsilon, 1-\tau_\varepsilon]$ into $N/2$ equidistant mesh intervals, and subdividing each of $[0,\tau_\varepsilon]$ and $[1-\tau_\varepsilon, 1]$ into $N/4$ equidistant mesh intervals. Set $\bar{\omega}_y = \bar{\omega}_x$ and let $\bar{\Omega}^N = \{(x_i,y_j)\}_{i,j=0}^N$ be the tensor product of $\bar{\omega}_x^N$ and $\bar{\omega}_y^N$. Set $\Omega^N = \bar{\Omega}^N \cap \Omega$ and $\partial\Omega^N = \bar{\Omega}^N \setminus \Omega^N$.

Set $h_i = x_i - x_{i-1}$ and $k_i = y_i - y_{i-1}$ for each i. Given a mesh function $\{v_{i,j}\}_{i,j=0}^N$, define the standard second-order central differencing operators

$$\delta_x^2 v_{i,j} := \frac{1}{\bar{h}_i}\left(\frac{v_{i+1,j}-v_{i,j}}{h_{i+1}} - \frac{v_{i,j}-v_{i-1,j}}{h_i}\right) \qquad \text{for } i = 1,\dots,N-1,$$

$$\delta_y^2 v_{i,j} := \frac{1}{\bar{k}_i}\left(\frac{v_{i,j+1}-v_{i,j}}{k_{i+1}} - \frac{v_{i,j}-v_{i,j-1}}{k_i}\right) \qquad \text{for } j = 1,\dots,N-1,$$

where $\bar{h}_i = (h_{i+1}+h_i)/2$ and $\bar{k}_i = (k_{i+1}+k_i)/2$. Set $\Delta^N v_{i,j} := (\delta_x^N + \delta_y^N)v_{i,j}$. Then we define the coupled difference operator as

$$(L^N\mathbf{U})_{i,j} = -\varepsilon^2\Delta^N\mathbf{U}_{i,j} + A(x_i,y_j)\mathbf{U}_{i,j}, \qquad \text{for } i = 1,\dots N-1, j = 1,\dots N-1. \tag{5.2}$$

To generate a numerical approximation of the solution to (1.1) solve the system of $2(N+1)^2$ linear equations

$$\begin{aligned} (L^N\mathbf{U})_{i,j} &= \mathbf{f}(x_i,y_j) \quad \text{for } (x_i,y_j) \in \Omega^N, \\ \mathbf{U}_{i,j} &= \mathbf{g}(x_i,y_i) \qquad\quad \text{for } (x_i,y_j) \in \partial\Omega^N. \end{aligned} \tag{5.3}$$

The analysis in this section is based on [3, Section 3].

By a *mesh function* $\mathbf{W} = (\mathbf{W})_{ij}$ we mean any vector-valued function that is defined at the points $\{(x_i,y_j)\}$ of the tensor-product Shishkin mesh of Section 5. For such functions, an inequality $\mathbf{V} \geq \mathbf{W}$ is understood to be an inequality on both components of $\mathbf{V}$ and $\mathbf{W}$, just as in the case of continuous functions in Section 4.

First, we derive discrete analogues of Lemmas 2.3 and 4.1. Recall the Shishkin mesh transition point defined in (5.1). In our analysis we shall assume that

$$\tau_\varepsilon = 2\sqrt{2}\,(\varepsilon/C_2)\ln N \tag{5.4}$$

as otherwise $\varepsilon^{-1} \leq C\ln N$ and a standard classical analysis then suffices.

Lemma 5.1. (discrete maximum principle) *Let $\mathbf{W}$ be a mesh function for which $L^N\mathbf{W} \geq \mathbf{0}$ on Ω^N and $\mathbf{W} \geq \mathbf{0}$ on $\partial\Omega^N$. Then $\mathbf{W} \geq \mathbf{0}$ on $\bar{\Omega}^N$.*

Proof. Apply to L^N and $\mathbf{W}$ the argument used to prove Lemma 2.3. ■

Set $C_3 = C_2/\sqrt{2}$.

Lemma 5.2. (discrete barrier functions) *Let* $\mathbf{C}_1$ *be as in Lemma 4.1. Then*

(i) $L^N\mathbf{C}_1 \geq \mathbf{1}$ *on* Ω^N*;*

(ii) there exists a mesh function $\mathbf{D}$ *such that for some* $\mathbf{C} > \mathbf{0}$ *we have* $\mathbf{D}_{ij} \geq \mathbf{C}e^{-C_3 y_j/\varepsilon}$ *on* $\bar{\Omega}^N$, $\mathbf{D}_{ij} \leq \mathbf{C}N^{-2}$ *when* $j \geq N/4$*, and* $(L^N\mathbf{D})_{ij} \geq \mathbf{0}$ *on* Ω^N*;*

(ii) there exists a mesh function $\mathbf{R}$ *such that for some* $\mathbf{C} > \mathbf{0}$ *we have* $\mathbf{R}_{ij} \geq \mathbf{C}e^{-C_3 x_i/\varepsilon}e^{-C_3 y_j/\varepsilon}$ *on* $\bar{\Omega}^N$*,* $\mathbf{R}_{ij} \leq \mathbf{C}N^{-2}$ *when* $i \geq N/4$ *or* $j \geq N/4$*, and* $(L^N\mathbf{R})_{ij} \geq \mathbf{0}$ *on* Ω^N*.*

Proof. To prove (i), choose $\mathbf{C}_1$ as in Lemma 4.1; the proof of that Lemma implies that $L^N\mathbf{C}_1 \geq \mathbf{1}$ on Ω^N.

For (ii),

$$L(e^{-C_3 y/\varepsilon}\mathbf{C}_1) = e^{-C_3 y/\varepsilon}(-C_3^2\mathbf{C}_1 + A\mathbf{C}_1) \geq e^{-C_3 y/\varepsilon}(\mathbf{1} - C_3^2\mathbf{C}_1)$$

which has positive components if we choose $C_2 > 0$ such that $C_2^2\gamma < 1$. Thus, setting $\mathbf{d} = Ce^{-C_2 y/\varepsilon}\mathbf{C}_1$ for a suitable constant C, we get (ii).

Towards proving (ii) and (iii), for all i and j set

$$\phi_{i,j} = \left[\prod_{r=1}^{N}\left(1 + \frac{k_r C_3}{\varepsilon}\right)\right]^{-1} \prod_{s=j+1}^{N}\left(1 + \frac{k_s C_3}{\varepsilon}\right).$$

Note that $\phi_{i,j}$ is independent of i. One has, for all i and j,

$$\phi_{i,j} = \prod_{r=1}^{j}\left(1 + \frac{k_r C_3}{\varepsilon}\right)^{-1} \geq \prod_{r=1}^{j}\exp(-k_r C_3/\varepsilon) = e^{-C_3 y_j/\varepsilon},$$

and when $j \geq N/4$,

$$\begin{aligned}
\phi_{i,j} = \prod_{r=1}^{j}\left(1 + \frac{k_r C_3}{\varepsilon}\right)^{-1} &\leq \prod_{r=1}^{N/4}\left(1 + \frac{k_r C_3}{\varepsilon}\right)^{-1} \\
&= \left(1 + \frac{4\tau_\varepsilon C_3}{\varepsilon N}\right)^{-N/4} \\
&= \left(1 + 8N^{-1}\ln N\right)^{-N/4} \\
&= \exp\left[-\frac{N}{4}\ln(1 + 8N^{-1}\ln N)\right] \\
&\leq CN^{-2},
\end{aligned}$$

by the easily-verified inequality $\ln(1+t) \geq t - t^2/2$ for all $t \geq 0$. Also, for $0 < j < N$,

$$\begin{aligned}
-\varepsilon^2(\delta_y^N\phi)_{i,j} &= -\frac{\varepsilon^2}{\bar{k}_j}\left(\frac{\phi_{i,j+1} - \phi_{i,j}}{k_{j+1}} - \frac{\phi_{i,j} - \phi_{i,j-1}}{k_j}\right) \\
&= -\frac{\varepsilon^2}{\bar{k}_j}\left[\prod_{r=1}^{N}\left(1 + \frac{k_r C_3}{\varepsilon}\right)\right]^{-1}\frac{C_3}{\varepsilon}\left[-\prod_{s=j+2}^{N}\left(1 + \frac{k_s C_3}{\varepsilon}\right) + \prod_{s=j+1}^{N}\left(1 + \frac{k_s C_3}{\varepsilon}\right)\right] \\
&= -\frac{C_3^2 k_{j+1}}{\bar{k}_j}\left[\prod_{r=1}^{N}\left(1 + \frac{k_r C_3}{\varepsilon}\right)\right]^{-1}\prod_{s=j+2}^{N}\left(1 + \frac{k_s C_3}{\varepsilon}\right) \\
&= -\frac{2C_3^2 k_{j+1}}{k_{j+1} + k_j}\phi_{i,j+1} \\
&\geq -2C_3^2\phi_{i,j} \\
&= -C_2^2\phi_{i,j}
\end{aligned}$$

and obviously $-\varepsilon^2(\delta_x^N\phi)_{i,j} = 0$.

Set $\mathbf{D}_{i,j} = \phi_{i,j}\mathbf{C}_1$ for all i and j. Then

$$(L^N\mathbf{D})_{i,j} \geq -\varepsilon^2(\delta_y^N\phi)_{i,j}\mathbf{C}_1 + \phi_{i,j}A\mathbf{C}_1 \geq (-C_2^2C_1+1)\phi_{i,j}\varepsilon \geq 0.$$

The other asserted inequalities follow from the above inequalities for $\phi_{i,j}$.

Similar calculations with

$$\mathbf{R}_{i,j} = \phi_{ij}\,\psi_{ij}\mathbf{C},$$

where

$$\psi_{ij} = \left[\prod_{r=1}^{N}\left(1+\frac{h_rC_3}{\varepsilon}\right)\right]^{-1}\prod_{s=i+1}^{N}\left(1+\frac{h_sC_3}{\varepsilon}\right),$$

yield (iii). ■

The mesh functions $\mathbf{D}$ and $\mathbf{R}$ of Lemma 5.2 will be used to deal with the boundary layer along $y = 0$ and the corner layer near the point $(0,0)$. One can define analogous mesh functions for the other boundary and corner layers in this problem.

Decompose the computed solution $\mathbf{U}$, analogously to (4.16), by setting

$$\mathbf{U} = \mathbf{V} + \sum_{k=1}^{4}\mathbf{W}_k + \sum_{k=1}^{4}\mathbf{Z}_k, \tag{5.5}$$

where the terms in this decomposition are defined by

$$\begin{cases} L^N\mathbf{V} = & \mathbf{f} \text{ in } \Omega^N, \\ \mathbf{V} = & \mathbf{v} \text{ on } \partial\Omega^N; \end{cases} \tag{5.6}$$

$$\begin{cases} L^N\mathbf{W}_k = & \mathbf{0} \text{ in } \Omega^N, \\ \mathbf{W}_k = & \mathbf{w}_k \text{ on } \partial\Omega^N, \end{cases} \quad \text{for } k = 1,2,3,4; \tag{5.7}$$

$$\begin{cases} L^N\mathbf{Z}_k = & \mathbf{0} \text{ in } \Omega^N, \\ \mathbf{Z}_k = & \mathbf{z}_k \text{ on } \partial\Omega^N, \end{cases} \quad \text{for } k = 1,2,3,4. \tag{5.8}$$

We begin with the smooth component $\mathbf{v}$.

Lemma 5.3. *There exists a positive constant C such that*

$$\|\mathbf{V}-\mathbf{v}\|_{\infty,\bar\Omega^N} \leq CN^{-2}\ln^2 N. \tag{5.9}$$

Proof. Consider the truncation error $L^N(\mathbf{V}-\mathbf{v})$. By the usual Taylor expansions about any point $(x_i,y_j)\in\Omega^N$, one obtains

$$|L^N(\mathbf{V}-\mathbf{v})(x_i,y_j)| \leq \begin{cases} \mathbf{C}\varepsilon^2\left(\bar h_i\|D_x^3\mathbf{v}\|_{\infty,\bar\Omega} + \bar k_j\|D_y^3\mathbf{v}\|_{\infty,\bar\Omega}\right) & \text{if } x_i \text{ or } y_j \text{ is a mesh transition point,} \\ \mathbf{C}\varepsilon^2\left(h_i^2\|D_x^4\mathbf{v}\|_{\infty,\bar\Omega} + k_j^2\|D_y^4\mathbf{v}\|_{\infty,\bar\Omega}\right) & \text{otherwise.} \end{cases}$$

Invoking (4.4), this yields

$$|L^N(\mathbf{V}-\mathbf{v})(x_i,y_j)| \leq \begin{cases} \mathbf{C}\varepsilon N^{-1} & \text{if } x_i \text{ or } y_j \text{ is a mesh transition point,} \\ \mathbf{C}N^{-2} & \text{otherwise.} \end{cases} \tag{5.10}$$

Define the barrier function

$$\mathbf{\Phi}(x,y) = \left[\frac{\tau_\varepsilon^2N^{-2}}{\varepsilon^2}\big(\theta(x)+\theta(y)\big) + N^{-2}\right]\mathbf{C}$$

where $\mathbf{C}$ is some sufficiently large positive multiple of the barrier function of Lemma 5.2(i) and the piecewise-linear function θ is defined by

$$\theta(t) = \begin{cases} t/\tau_\varepsilon & \text{when } 0 \leq t \leq \tau_\varepsilon, \\ 1 & \text{when } \tau_\varepsilon \leq t \leq 1-\tau_\varepsilon, \\ (1-t)/\tau_\varepsilon & \text{when } 1-\tau_\varepsilon \leq t \leq 1. \end{cases}$$

A quick calculation shows that $-\delta_x^2\theta(x_i) \geq 2N/\tau_\varepsilon$ if x_i is a mesh transition point, while $-\delta_x^2\theta(x_i) = 0$ for other mesh points. There is a similar result for $-\delta_y^2\theta(y_j)$. From these observations, Lemma 5.2(i) and (5.10) , we can apply Lemma 5.1 to $\mathbf{\Phi} \pm (\mathbf{V} - \mathbf{v})$ to get

$$\|\mathbf{V} - \mathbf{v}\|_{\infty,\bar{\Omega}^N} \leq \|\mathbf{\Phi}\|_{\infty,\bar{\Omega}^N} \leq CN^{-2}\ln^2 N.$$

■

Next, consider the boundary layer functions $\mathbf{w}_k$, for $k = 1, 2, 3, 4$.

Lemma 5.4. *There exists a positive constant C such that*

$$\|\mathbf{W}_k - \mathbf{w}_k\|_{\infty,\bar{\Omega}^N} \leq CN^{-2}\ln^2 N \quad \textit{for } k = 1, 2, 3, 4. \tag{5.11}$$

Proof. We prove the result only for $k = 1$; the other boundary layers are similar. Set $\Omega_y^N = \{(x_i, y_j) \in \Omega^N : y_j > \tau_\varepsilon\}$. Inequality (4.12c) and definition (5.4) imply that $\|\mathbf{w}_1\|_{\infty,\bar{\Omega}_y^N} \leq CN^{-2}$. Again appealing to (4.12c) and to (5.7), we see that $|\mathbf{W}_1(x_i, y_j)| \leq Ce^{-C_2 y_j/\varepsilon}\mathbf{1}$ on $\partial\Omega^N$. As we also have $L^N\mathbf{W}_1 = \mathbf{0}$ on $\bar{\Omega}^N$, it follows from Lemma 5.2(ii) and the discrete maximum principle (Lemma 5.1) that $|\mathbf{W}_1(x_i, y_j)| \leq \mathbf{D}_{ij}$ on $\bar{\Omega}^N$, and consequently $|\mathbf{W}_1(x_i, y_j)| \leq \mathbf{C}N^{-2}$ on $\bar{\Omega}_y^N$. Thus a triangle inequality yields

$$\|\mathbf{W}_1 - \mathbf{w}_1\|_{\infty,\bar{\Omega}_y^N} \leq CN^{-2}. \tag{5.12}$$

To bound $\mathbf{W}_1 - \mathbf{w}_1$ on $\bar{\Omega}^N \setminus \Omega_y^N$, note first that (5.12) and the definition of $\mathbf{W}_1$ yield a bound for $\mathbf{W}_1 - \mathbf{w}_1$ on the boundary of this region, then consider the truncation error when $0 < i < N$ and $0 < j < N/4$. By Taylor expansions we obtain

$$|L^N(\mathbf{W}_1 - \mathbf{w}_1)(x_i, y_j)| \leq \begin{cases} \mathbf{C}\varepsilon^2\left(\bar{h}_i\|D_x^3\mathbf{w}_1\|_{\infty,\bar{\Omega}} + k_j^2\|D_y^4\mathbf{w}_1\|_{\infty,\bar{\Omega}}\right) & \text{if } x_i \text{ is a mesh transition point,} \\ \mathbf{C}\varepsilon^2\left(h_i^2\|D_x^4\mathbf{w}_1\|_{\infty,\bar{\Omega}} + k_j^2\|D_y^4\mathbf{w}_1\|_{\infty,\bar{\Omega}}\right) & \text{otherwise.} \end{cases}$$

Invoking (4.12a) and (4.12b) yields

$$|L^N(\mathbf{W}_1 - \mathbf{w}_1)(x_i, y_j)| \leq \begin{cases} \mathbf{C}N^{-2}\ln^2 N + \mathbf{C}\varepsilon N^{-1} & \text{if } x_i \text{ is a mesh transition point,} \\ \mathbf{C}N^{-2}\ln^2 N & \text{otherwise.} \end{cases}$$

Now, similarly to the proof of Lemma 5.3, we can use the barrier function $\left[\tau_\varepsilon^2 N^{-2}\varepsilon^{-2}\theta(x) + N^{-2}\ln^2 N\right]\mathbf{C}$ and the discrete maximum principle of Lemma 5.1, applied on $\Omega^N \setminus \Omega_y^N$, to prove $\|\mathbf{W}_1 - \mathbf{w}_1\|_{\infty,\bar{\Omega}^N\setminus\Omega_y^N} \leq CN^{-2}\ln^2 N$.

Inequality (5.11) now follows. ■

Finally, we come to the corner layer functions $\mathbf{z}_k$, for $k = 1, 2, 3, 4$.

Lemma 5.5. *There exists a positive constant C such that*

$$\|\mathbf{Z}_k - \mathbf{z}_k\|_{\infty,\bar{\Omega}^N} \leq CN^{-2}\ln^2 N \quad \textit{for } k = 1, 2, 3, 4. \tag{5.13}$$

Proof. The result will be proved only for $k = 1$ as the other corner layers are similar. The structure of the analysis is very similar to that of Lemma 5.4. First one uses the decay of $\mathbf{z}_1$, given by (4.15), to guarantee that $\|\mathbf{z}_1\|_{\infty,\bar{\Omega}^N\setminus\Omega_1^N} \leq CN^{-2}$, where Ω_1^N is the fine-mesh neighbourhood of the corner (0,0) defined by $\{(x_i, y_j) \in \bar{\Omega}^N : x_i < \tau_\varepsilon \text{ and } y_j < \tau_\varepsilon\}$. Next, as $|\mathbf{Z}_1(x_i, y_j)| \leq Ce^{-C_2 x_i/\varepsilon}e^{-C_2 y_j/\varepsilon}\mathbf{1}$ on $\partial\Omega^N$, and $L^N\mathbf{Z}_1 = \mathbf{0}$ on Ω^N, it follows from Lemmas 5.2(iii) and 5.1 that $|\mathbf{Z}_1(x_i, y_j)| \leq \mathbf{R}_{ij} \leq \mathbf{C}N^{-2}$ on $\bar{\Omega}^N \setminus \Omega_1^N$. Now a triangle inequality yields

$$\|\mathbf{Z}_1 - \mathbf{z}_1\|_{\infty,\bar{\Omega}^N\setminus\Omega_1^N} \leq CN^{-2}. \tag{5.14}$$

To bound $\mathbf{Z}_1 - \mathbf{z}_1$ on $\bar{\Omega}_1^N$, for $0 < i, j < N/4$ we have

$$|L^N(\mathbf{Z}_1 - \mathbf{z}_1)(x_i, y_j)| \leq \mathbf{C}\varepsilon^2\left(h_i^2\|D_x^4\mathbf{w}_1\|_{\infty,\bar{\Omega}} + k_j^2\|D_y^4\mathbf{w}_1\|_{\infty,\bar{\Omega}}\right) \leq CN^{-2}\ln^2 N$$

by (4.14). Now use the barrier function $(N^{-2}\ln^2 N)\mathbf{C}$ and a discrete maximum principle on $\bar{\Omega}_1^N$ to show that $\|\mathbf{Z}_1 - \mathbf{z}_1\|_{\infty,\bar{\Omega}_1^N} \leq CN^{-2}\ln^2 N$. Inequality (5.13) then follows. ■

Using a triangle inequality to combine Lemmas 5.3–5.5, we finally obtain our convergence result:

Theorem 5.1. *Let* $\mathbf{u}$ *be the solution to (1.1) and* $\mathbf{U}$ *the solution to the discrete problem (5.2) on the Shishkin mesh* Ω^N *of Section 5. Then there exists a positive constant* C *such that*

$$\|\mathbf{u}-\mathbf{U}\|_{\infty,\bar{\Omega}^N} \leq C(N^{-1}\ln N)^2.$$

6 Numerical Results

We now present numerical results in support of Theorem 5.1. In particular, we wish to show that, for interesting examples, the method is ε-uniformly convergent. We also want to demonstrate that the theorem is sharp: the observed rate of convergence is indeed almost second order.

The finite difference method (5.3) requires the solution of a linear system of $M(N+1)^2$ equations which may be solved directly. However, it is often useful to resolve the system by iteratively solving the system in analogy to Lemma 2.2. Define the decoupled discrete operators:

$$\big(L_m^N W\big)(x_i,y_j) := -\varepsilon^2\Delta^N W(x_i,y_j) + \big(a_{mm}W\big)(x_i,y_j), \quad \text{for } (x_i,y_j)\in\Omega^N \text{ and } m=1,2,\dots,M.$$

Let $\mathbf{U}$ be a solution of (5.3). Define the sequence of vector-valued mesh functions $\{\mathbf{U}^{[k]}\}_{k=0}$: let $\mathbf{U}^{[0]}$ be a mesh function such that $\mathbf{U}^{[0]}(x_i,y_j)=\mathbf{U}(x_i,y_j)$ for $(x_i,y_j)\in\partial\Omega^N$, and for $k=1,2,\dots,$ solve

$$\begin{aligned}\big(L_m^N U_m^{[k]}\big)(x_i,y_j) &= f_m(x_i,y_j) - \sum_{n<m}\big(a_{mn}U_n^{[k]}\big)(x_i,y_j) - \sum_{n>m}\big(a_{mn}U_n^{[k-1]}\big)(x_i,y_j) \text{ on } \Omega^N,\\ \big(U_m^{[k]}\big)(x_i,y_j) &= \big(U_m\big)(x_i,y_j) \text{ on } \partial\Omega^N.\end{aligned}$$

It is easy to show that $\mathbf{U}^{[k]}$ converges to $\mathbf{U}$ by following the analysis of Lemma 2.2 or adapting the arguments of [14, Lemma 8]. Among the advantages of this approach are the requirement of less memory by the computing system and the ability to reuse a solver previously written for the uncoupled elliptic problem.

We will consider several test problems. In all cases the true solution $\mathbf{u}$ is unavailable so we estimate the error numerically using the *two-mesh difference* approach; see [5, §8.2]. The two-mesh difference, for fixed N and ε, is defined as

$$\mathbf{D}_\varepsilon^N := \max_{0\leq i,j\leq N}|\tilde{\mathbf{U}}_{2i,2j}^{2N}-\mathbf{U}_{i,j}^N|, \qquad D_\varepsilon^N := \max_{k=1,\dots,M}\{D_{\varepsilon,k}^N\}$$

where $\tilde{U}^N$ is the solution computed on the mesh obtained by bisecting Ω^N in both the x- and y-directions. The ε-uniform two-mesh difference is computed as

$$D^N := \max_{\varepsilon^2=1,10^{-1},\dots,10^{-12}} D_\varepsilon^N.$$

The rate of convergence is estimated by $\rho_\varepsilon^N = \log_2(D_\varepsilon^N/D_\varepsilon^{2N})$, with the ε-uniform rate given as

$$\rho^N := \min_{\varepsilon^2=1,10^{-1},\dots,10^{-12}} \rho_\varepsilon^N.$$

Then, in order to show that the rate of convergence obtained is indeed almost second order as proved in Theorem 5.1, we compute

$$C^N := D^N(N/\ln N)^2.$$

If the convergence rate is $O(N^{-2}/\ln^2 N)$, then C_N should be independent of N.

Example 6.1 (Constant coefficients)**.** *Our first test case is (1.1) with*

$$A=\begin{pmatrix}3 & -5\\ -1 & 4\end{pmatrix}, \qquad \mathbf{f}=\begin{pmatrix}1+x\\ 1+y\end{pmatrix},$$

and homogeneous boundary conditions

$$\mathbf{u}(x,y)=\mathbf{0} \quad \textit{for } (x,y)\in\partial\Omega.$$

Note that A satisfies conditions (2.1a) and (2.1b), but not (2.1c). Nevertheless the weaker condition (2.4) is satisfied, and the mesh transition point is chosen as in Remark 5.1.

Table 1 below gives results for Example 6.1 for various values of ε and N. As proved in Theorem 5.1 the method is uniformly convergent with respect to ε. Furthermore, the ε-uniform rate of convergence ρ_ε^N is clearly almost second order.

	N				
ε^2	32	64	128	256	512
10^0	6.77e-05	1.70e-05	4.24e-06	1.06e-06	2.65e-07
10^{-1}	7.34e-04	1.84e-04	4.60e-05	1.15e-05	2.88e-06
10^{-2}	5.40e-03	1.38e-03	3.45e-04	8.64e-05	2.16e-05
10^{-3}	4.68e-02	1.33e-02	3.47e-03	8.79e-04	2.20e-04
10^{-4}	1.42e-01	9.85e-02	3.04e-02	8.63e-03	2.20e-03
10^{-5}	1.43e-01	1.04e-01	4.38e-02	1.63e-02	5.35e-03
10^{-6}	1.43e-01	1.04e-01	4.38e-02	1.63e-02	5.35e-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-12}	1.43e-01	1.04e-01	4.38e-02	1.63e-02	5.36e-03
D^N	1.43e-01	1.04e-01	4.38e-02	1.63e-02	5.36e-03
ρ^N	0.46	1.25	1.43	1.61	
C^N	12.20	24.66	30.49	34.77	36.08

Table 1: Errors and rates of convergence in the computed solution to Example 6.1.

Example 6.2. *We consider problem (1.1) with the variable coefficient matrix*

$$A = \begin{pmatrix} 1 + x(1-y) & -x^{(y+1)}/3 \\ -1/(2+2xy) & 2\cos(\pi x/3) \end{pmatrix}.$$

The boundary conditions are

$$\mathbf{u}(x,y) = \begin{pmatrix} (1-x)(1-y) \\ 2+3xy \end{pmatrix} \qquad \text{for } (x,y) \in \partial\Omega.$$

To satisfy Assumption 3.1 we choose $\mathbf{f}$ *to be the bilinear function that equals* $A\mathbf{u}$ *at the corners of the domain.*

The numerical results are given in Table 2 below and again support the theoretical findings of Theorem 5.1.

	N				
ε^2	32	64	128	256	512
10^0	7.51e-06	1.89e-06	4.72e-07	1.18e-07	2.95e-08
10^{-1}	1.53e-04	3.84e-05	9.60e-06	2.40e-06	6.00e-07
10^{-2}	1.70e-03	4.38e-04	1.10e-04	2.75e-05	6.88e-06
10^{-3}	1.48e-02	4.11e-03	1.08e-03	2.73e-04	6.85e-05
10^{-4}	3.07e-02	2.12e-02	9.17e-03	2.67e-03	6.80e-04
10^{-5}	3.07e-02	2.12e-02	9.17e-03	3.19e-03	1.03e-03
10^{-6}	3.08e-02	2.12e-02	9.17e-03	3.19e-03	1.03e-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-12}	3.08e-02	2.12e-02	9.17e-03	3.19e-03	1.03e-03
D^N	3.08e-02	2.12e-02	9.17e-03	3.19e-03	1.03e-03
ρ^N	0.53	1.21	1.52	1.63	
C^N	2.62	5.03	6.38	6.80	6.93

Table 2: Errors and rates of convergence in the computed solution to Example 6.2.

Imitating [2, §4], we now investigate the effects on the rates of convergence of violating the compatibility conditions that we assumed in Assumption 3.1.

Example 6.3. *We take the same coefficient matrix A and boundary conditions as in Example 6.2, but $\mathbf{f}$ as in Example 6.1. Thus although the boundary conditions are continuous, Assumption 3.1 is not satisfied.*

The results are shown in Table 3. The rates are similar to those observed in Examples 6.1 and 6.2.

	N				
ε^2	32	64	128	256	512
10^0	9.14e-05	2.29e-05	5.73e-06	1.43e-06	3.58e-07
10^{-1}	7.63e-04	1.92e-04	4.82e-05	1.20e-05	3.01e-06
10^{-2}	7.51e-03	1.96e-03	4.95e-04	1.24e-04	3.11e-05
10^{-3}	6.15e-02	1.81e-02	4.93e-03	1.26e-03	3.18e-04
10^{-4}	1.43e-01	9.21e-02	3.90e-02	1.21e-02	3.15e-03
10^{-5}	1.46e-01	9.30e-02	3.92e-02	1.44e-02	4.79e-03
10^{-6}	1.47e-01	9.34e-02	3.92e-02	1.44e-02	4.80e-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-12}	1.48e-01	9.35e-02	3.93e-02	1.45e-02	4.80e-03
D^N	1.48e-01	9.35e-02	3.93e-02	1.45e-02	4.80e-03
ρ^N	0.64	1.24	1.44	1.59	
C^N	12.59	22.14	27.34	30.80	32.34

Table 3: Errors and rates of convergence in the computed solution to Example 6.3

To conclude our experiments on the effect that the violation of compatibility conditions has on the rate of convergence of the numerical scheme, we consider a case where the boundary data are discontinuous.

Example 6.4. *Take A and f as in Example 6.3, but with the boundary conditions*

$$u_1(x,y) = \begin{cases} 1 & \textit{for } y = 1 \\ 0 & \textit{otherwise} \end{cases} \qquad \textit{and} \qquad u_2(x,y) = \begin{cases} 1 & \textit{for } x = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

The results are shown in Table 4. For large ε, no convergence is observed (and hence ρ^N is omitted from the table). For small ε, as can be seen from the final row of the table, the rate of convergence is significantly less than almost second order.

	N				
ε^2	32	64	128	256	512
10^0	4.90e-03	4.89e-03	4.88e-03	4.88e-03	4.88e-03
10^{-1}	5.07e-03	4.93e-03	4.89e-03	4.88e-03	4.88e-03
10^{-2}	6.89e-03	5.37e-03	5.00e-03	4.91e-03	4.89e-03
10^{-3}	2.96e-02	1.03e-02	6.20e-03	5.19e-03	4.96e-03
10^{-4}	9.53e-02	4.87e-02	1.84e-02	8.38e-03	5.70e-03
10^{-5}	9.82e-02	4.95e-02	1.87e-02	9.17e-03	6.16e-03
10^{-6}	9.91e-02	4.97e-02	1.88e-02	9.18e-03	6.16e-03
10^{-7}	9.94e-02	4.98e-02	1.89e-02	9.19e-03	6.17e-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-12}	9.96e-02	4.98e-02	1.89e-02	9.19e-03	6.17e-03
D^N	9.96e-02	4.98e-02	1.89e-02	9.19e-03	6.17e-03
C^N	8.49	11.80	13.14	19.58	41.53

Table 4: Errors and rates of convergence in the computed solution to Example 6.4

Most studies in the literature of coupled systems of singularly perturbed differential equations consider just two equations. In our analyses we have presented theoretical results for the general case of M equations, so we take $M = 3$ in our final example.

Example 6.5 ($M = 3$). *Let*

$$A \equiv \begin{pmatrix} 6 & -1 & -1 \\ -2 & 4 & -1 \\ -1 & -2 & 5 \end{pmatrix}$$

and assume homogeneous boundary conditions

$$\mathbf{u}(x, y) = \mathbf{0} \quad \textit{for } (x, y) \in \partial\Omega.$$

The results are presented in Table 5 below and support Theorem 5.1: the finite difference method (5.3) on the mesh described in §5 yields a numerical solution that converges at a rate that is independent of ε and almost second order.

	N				
ε^2	32	64	128	256	512
10^{0}	8.37e-05	2.10e-05	5.24e-06	1.31e-06	3.28e-08
10^{-1}	6.48e-04	1.64e-04	4.10e-05	1.03e-05	2.57e-06
10^{-2}	5.97e-03	1.61e-03	4.10e-04	1.03e-04	2.59e-05
10^{-3}	4.06e-02	1.37e-02	3.93e-03	1.02e-03	2.60e-04
10^{-4}	6.25e-02	4.39e-02	2.02e-02	7.41e-03	2.49e-03
10^{-5}	6.27e-02	4.40e-02	2.03e-02	7.43e-03	2.53e-03
10^{-6}	6.28e-02	4.40e-02	2.03e-02	7.43e-03	2.53e-03
⋮	⋮	⋮	⋮	⋮	⋮
10^{-12}	6.28e-02	4.40e-02	2.03e-02	7.43e-03	2.53e-03
D^N	6.28e-02	4.40e-02	2.03e-02	7.43e-03	2.53e-03
ρ^N	0.51	1.12	1.45	1.55	
C^N	5.36	10.43	14.11	15.84	17.07

Table 5: Numerical results for a system of 3 equations

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