# Commensurations and Subgroups of Finite Index of Thompson's Group $F$

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We determine the abstract commensurator $Com(F)$ of Thompson's group $F$ and describe it in terms of piecewise linear homeomorphisms of the real line. We show $Com(F)$ is not finitely generated and determine which subgroups of finite index in $F$ are isomorphic to $F$. We also show that the natural map from the commensurator group to the quasi-isometry group of $F$ is injective.

## Introduction

Thompson's groups have been extensively studied since their introduction by Thompson in the 1960s, despite the fact that Thompson's account [7] appeared only in 1980. They have provided examples of infinite finitely presented simple groups, as well as some other interesting counterexamples in group theory (see for example, Brown and Geoghegan [3]). Cannon, Floyd and Parry [4] give an excellent introduction to Thompson's groups where many of the basic results used below are proven carefully.

Automorphisms for Thompson's group $F$ were studied by Brin [2], where a key theorem by McCleary and Rubin [6] is used to realize each automorphisms as conjugation by a piecewise linear map. Here, we generalize from automorphisms to commensurations, which are isomorphisms between two subgroups of finite index. These form a group (under a natural equivalence relation involving passing to smaller yet still finite-index subgroups), called the commensurator group.

We classify finite-index subgroups of $F$, and then we extend Brin's results from automorphisms to commensurations, again realizing every commensuration as conjugation by a piecewise linear homeomorphism of the real line. These maps exhibit a particular structure, satisfying an affinity condition in the neighborhood of $\infty$ which we use to find the algebraic structure of the commensurator of $F$.

Commensurators have proven to be an effective tool for investigating quasi-isometries of a group to itself, and for effectively analyzing rigidity, particularly of lattices. In the case of $F$, the only quasi-isometries of $F$ known previously were automorphisms. This paper provides a wide array of examples of quasi-isometries, since all commensurations are quasi-isometries, and we prove in Section 5 that the commensurator group embeds into the quasi-isometry group in the case of $F$.

Our approach is algebraic, but we note that elements of the commensurator of $F$ can be represented by marked, infinite, eventually periodic, binary tree pair diagrams. We also note that recently Bleak and Wassink [1] have independently described the finite-index subgroups of $F$, using different methods.

The paper is organized as follows. In Section 1 we give the necessary definitions, and in Section 2 the first basic results for the finite-index subgroups of $F$. In Section 3 the main result about the commensurator is stated and proved, and in Section 4 its algebraic structure is given. The proof of the embedding of the commensurator group into the quasi-isometry group is given in Section 5.

### Acknowledgements

## 1 Definitions

Let $P$ denote the group of all homeomorphisms $f$ from $\mathbb{R}$ to itself that

(1) are piecewise linear with a discrete (but possibly infinite) set of breakpoints (discontinuities of the derivative of $f$),

(2) use only slopes that are integral powers of $2$,

(3) have their breakpoints in the set $\mathbb{Z}[\frac{1}{2}]$ and

(4) satisfy $f(\mathbb{Z}[\frac{1}{2}]) \subset \mathbb{Z}[\frac{1}{2}]$.

It is easy to check that each element $f$ of $P$ actually satisfies $f(\mathbb{Z}[\frac{1}{2}]) = \mathbb{Z}[\frac{1}{2}]$ and that $P$ has a subgroup of index two which contains only the order preserving elements. We denote this subgroup by $P_+$. The quotient $P/P_+$ is generated by the image of the homeomorphism $\tau\colon t \mapsto -t$.

Let $f \in P$. We call $f$ *integrally affine* if $f(t) = \varepsilon t + p$ for some integer $p$ and $\varepsilon \in \{\pm 1\}$. We say $f$ is *periodically affine* if $f(t + p) = f(t) + q$ for some non-zero $p, q \in \mathbb{R}$ and *integrally periodically affine* if $p$ and $q$ are integers. Note that all integrally affine maps are integrally periodically affine with $q = \pm p$ depending on whether $f$ is in $P_+$ or not.

When $\mathcal{P}$ is any of the above properties, then we call $f$ *eventually* $\mathcal{P}$ if $f$ satisfies $\mathcal{P}$ for all $t \in \mathbb{R}$ with $|t| > M$ for some $M > 0$; here $|t|$ denotes the absolute value of $t$. For example, $f \in P_+$ is eventually integrally affine if there exist $l, r \in \mathbb{Z}$, $M \in \mathbb{R}$, $M > 0$, so that $f(t) = t + r$ for all $t > M$ and $f(t) = t + l$ for all $t < -M$. Notice that $l$ and $r$ may well be different.

It is well-known that Thompson's group $F$ is isomorphic to the subgroup of $P_+$ consisting of all eventually integrally affine elements (see [4]). It is easy to see that the commutator subgroup $F'$ of $F$ consists of all eventually trivial elements of $P_+$ (those where eventually $f(t) = t$). This group is denoted by $BPL_2(\mathbb{R})$ by Brin [2], where $B$ stands for bounded support.

## 2 Finite-index Subgroups of $F$

Let $f$ be an element of $F$. Since $f$ is eventually integrally affine, there are two integers $l, r$ and a real number $M > 0$ such that $f(t) = t + r$ for $t > M$ and $f(t) = t + l$ for $t < -M$. The two numbers $l$ and $r$ are precisely the two components of the image of $f$ in $\mathbb{Z} \times \mathbb{Z}$ under the abelianization map. The subgroups of finite index of $F$ are in one-to-one correspondence with those of its abelianization $\mathbb{Z} \times \mathbb{Z}$ by the following result.

**Proposition 2.1** *Let $H$ be a subgroup of $F$ of finite index. Then $H$ contains $F'$, the commutator subgroup of $F$, and hence $H$ is normal in $F$. Moreover, $H' = F'$.*

**Proof** Since $F$ is finitely generated, $H$ has only finitely many conjugates in $F$ and the intersection of all of them, $K$ say, is normal and of finite index in $F$. We consider $K \cap F'$, which is thus normal and of finite index in $F'$. Hence, since $F'$ is simple and infinite, we conclude that $K \cap F' = F'$ and $F' \subset K \subset H$.

Hence $H$ is normal in $F$. The final claim follows from the fact that $H'$ is contained in $F'$ but also characteristic in $H$ and hence normal in $F$, whence $F' \subset H'$.                    □

From this fact we deduce that the finite-index subgroups of $F$ are in bijection with those of $\mathbb{Z} \times \mathbb{Z}$. There is a distinguished family among these—the subgroups $p\mathbb{Z} \times q\mathbb{Z}$. We denote by $[p, q]$, $p, q \in \mathbb{Z}$, the preimage in $F$ under the abelianization homomorphism of the subgroup $p\mathbb{Z} \times q\mathbb{Z}$ of $\mathbb{Z} \times \mathbb{Z}$. Thus $F = [1, 1]$ and $F' = [0, 0]$.

## 3   The Commensurator Group

As mentioned before, a *commensuration* of a group $G$ is an isomorphism $\alpha \colon A \to B$, where $A$ and $B$ are subgroups of $G$ of finite index. Two commensurations $\alpha$ and $\beta$ are equivalent if they agree on some subgroup of finite index in $G$. In view of this, the product $\beta \circ \alpha$ of two commensurations

$$\alpha \colon A \to B \quad \text{and} \quad \beta \colon C \to D$$

is defined on $\alpha^{-1}(B \cap C)$. The set of all commensurations of $G$ modulo the above equivalence relation, together with this composition, forms a group called the *commensurator of $G$* which we denote by $\mathrm{Com}(G)$. If $G$ is a subgroup of the group $H$, then the (relative) commensurator of $G$ in $H$, $\mathrm{Com}_H(G)$, consists of all elements $h$ of $H$ for which $G \cap G^h$ has finite index in both $G$ and $G^h$; here $G^h = h^{-1}Gh$.

The main result of this paper is the following.

**Theorem 3.1**   *The commensurator of $F$ is isomorphic to $\mathrm{Com}_P(F)$, which consists of all eventually integrally periodically affine elements (of $P$).*

The strategy of the proof is to find a large group where $F$ is a subgroup, and in such a way that every commensuration can be seen as a conjugation by an element of the large group. The group $P$ plays this role in the case of $F$.

In order to explain this strategy, we need some definitions and one of the main results of McCleary and Rubin [6]. Let $(L, <)$ be a dense linear order. By *interval* we mean a nonempty open interval. A subgroup $G$ of $\mathrm{Aut}(L)$ is *locally moving* if for every interval $I$ there exists a nontrivial element $g \in G$ which acts as the identity on $L \setminus I$. Finally, $G$ is *$n$-interval-transitive* if for every pair of sequences of intervals $I_1 < \cdots < I_n$ and $J_1 < \cdots < J_n$ there exists $g \in G$ such that $I_k^g \cap J_k \neq \varnothing$ for $1 \leq k \leq n$. Below, $\overline{L}$ denotes the Dedekind completion of $L$ which is assumed to have no endpoints.

**Theorem 3.2** (McCleary–Rubin [6])  *Assume $(L_i, <)$ is a dense linear order without endpoints and let $G_i \subset \mathrm{Aut}(L_i)$ be locally moving and $2$-interval transitive, $i = 1, 2$. Suppose that $\alpha \colon G_1 \to G_2$ is an isomorphism. Then there is a monotonic bijection $\tau \colon \overline{L}_1 \to \overline{L}_2$ which induces $\alpha$, that is, $g^\alpha = \tau^{-1} g \tau$ for every $g \in G_1$; and $\tau$ is unique.*

Being locally moving and having 2-interval transitivity are local properties in the sense that a group inherits these from any of its subgroups.

**Proof of Theorem 3.1**  View $\mathbb{Z}[\frac{1}{2}]$ as a dense linear order and $F$ as the eventually integrally affine elements of $P_+$. Let $\alpha \colon A \to B$ be a commensuration of $F$. By Proposition 2.1, both $A$ and $B$ contain $F'$ which is (obviously) locally moving and 2-interval transitive (see [2, Lemma 2.1]). So Theorem 3.2 tells us that $\alpha$ is induced by conjugation with a unique element of $\mathrm{Homeo}(\mathbb{R})$. This yields an injective homomorphism $\Psi \colon \mathrm{Com}(F) \to \mathrm{Homeo}(\mathbb{R})$.

Next, we show that the image of $\Psi$ is in fact contained in $P$. By Proposition 2.1, each commensuration of $F$ induces an automorphism of $F'$. In other words, the image of $\Psi$ is contained in $N_{\mathrm{Homeo}(\mathbb{R})}(F')$, the normalizer of $F'$ in $\mathrm{Homeo}(\mathbb{R})$. But this normalizer is equal to $P$ by Theorem 1 of Brin [2]. The existence and uniqueness statements in Theorem 3.2 now imply that $\Psi$ is an isomorphism between $\mathrm{Com}(F)$ and $\mathrm{Com}_P(F)$, which proves the first part of Theorem 3.1.

Let $\alpha \in \mathrm{Com}(F)$ and choose positive integers $p$ and $q$ so large that $\alpha$ is defined on the subgroup $[p, q]$, that is $[p, q]^\alpha$, the image of $[p, q]$ under $\alpha$, is contained in $F$. By what was said above, we can view $\alpha$ as conjugation by an element of $P$. So for $f \in [p, q]$ we find $f^\alpha = \alpha^{-1} f \alpha$ to be eventually integrally affine. Suppose for a moment that $\alpha$ is order preserving and that $f(t) = t + kq$ for $t \gg 0$, where $k \in \mathbb{Z}$. Then

$$f^\alpha(t) = (\alpha \circ f \circ \alpha^{-1})(t) = \alpha(f(\alpha^{-1}(t))) = \alpha(\alpha^{-1}(t) + kq) = t + r$$

must hold for some $r \in \mathbb{Z}$. In other words, $\alpha^{-1}(t + r) = \alpha^{-1}(t) + s$ for some integers $r$ and $s$ and all $t \gg 0$. Since $f$ was arbitrary, we may assume that $k \neq 0$, which implies that $s \neq 0$, and hence also $r \neq 0$. Therefore $\alpha^{-1}$, and hence $\alpha$, must be integrally periodically affine near infinity. A similar calculation holds for $t \ll 0$ and also when $\alpha$ is order reversing. Consequently, each commensuration of $F$ must be eventually integrally periodically affine.

It remains to show that each eventually integrally periodically affine $\beta \in P$ induces a commensuration of $F$ by conjugation. Suppose $\beta(t + p) = \beta(t) + q$ for $t \gg 0$ and $\beta(t + p') = \beta(t) + q'$ for $t \ll 0$, with $p, q, p', q' \in \mathbb{Z} \setminus \{0\}$. Let $U = [p', p]$ if $\beta$ is

order preserving and set $U = [p, p']$ otherwise. Then for $f \in U$, we have

$$f^\beta(t) = \begin{cases} \beta(\beta^{-1}(t) + kp) = t + kq, & t \gg 0 \\ \beta(\beta^{-1}(t) + k'p') = t + k'q', & t \ll 0 \end{cases}$$

where $k, k' \in \mathbb{Z}$ depend on $f$. Together with a similar argument for $\beta^{-1}$ one easily sees that $U^\beta = [q', q]$ or $[q, q']$, depending on whether $\beta$ is order preserving or not. Theorem 3.1 is thus established.                                                                                      $\square$

We immediately obtain the following corollaries from this result.

**Corollary 3.3** *A subgroup $U$ of $F$ of finite index is isomorphic to $F$ if and only if $U = [p, q]$ for some positive integers $p$ and $q$.*

**Proof** Suppose $U$ is a subgroup of finite index in $F$. If $U$ is isomorphic to $F$, then there exists an eventually integrally periodically affine $\alpha \in P$ with $F^\alpha = U$ and calculations as above show that $U$ must be of the form $[p, q]$. On the other hand, the final paragraph of the proof of the theorem read with $p = p' = 1$ shows that $[q', q]$ is isomorphic to $F$ for every choice of positive integers $q$ and $q'$. This completes the proof.                                                                                      $\square$

Finally, since each subgroup of finite index in $F$ contains $[p, q]$ for some positive integers $p$ and $q$ by Proposition 2.1, we have the following results.

**Corollary 3.4** *Every finite-index subgroup of $F$ is virtually $F$.*

**Corollary 3.5** *A group is commensurable with $F$ if and only if it is a finite extension of $F$.*

## 4 The Structure of $\mathrm{Com}(F)$

Descriptions of elements of $\mathrm{Com}(F)$ as conjugations in $P$ allow us to study its structure as a group. An element $\alpha$ of $\mathrm{Com}(F)$ is eventually integrally periodically affine, so there exist positive integers $p, p', q, q'$ and a real number $M$ such that

$$\alpha(t + p) = \alpha(t) + q, \text{ for } t > M$$
$$\alpha(t + p') = \alpha(t) + q', \text{ for } t < -M.$$

We need a lemma about affine functions, whose proof is elementary and left to the reader.

**Lemma 4.1** *Let $f\colon \mathbb{R} \to \mathbb{R}$ be an integrally periodically affine map, and assume that there are integers $i, i', j, j'$ such that for all $t \in \mathbb{R}$ we have*

$$f(t+i) = f(t)+j \quad \text{and} \quad f(t+i') = f(t)+j'.$$

*Then we have*

$$f(t+r) = f(t)+s,$$

where
$$r = \gcd(i, i') \quad \text{and} \quad s = \gcd(j, j').$$

*Furthermore, we have*

$$\frac{i}{j} = \frac{i'}{j'}.$$

From this lemma, we see that the integers $p, p', q, q'$ for element of $\mathrm{Com}(F)$ depend only on the element.

We recall that $\mathrm{Com}(F)$ has a subgroup of index 2, denoted $\mathrm{Com}^+(F)$, formed by the commensurations induced by conjugations by piecewise-linear maps which preserve the orientation of $\mathbb{R}$.

**Proposition 4.2** *There exists a surjective homomorphism $\Phi\colon \mathrm{Com}^+(F) \to \mathbb{Q}^* \times \mathbb{Q}^*$ defined by*

$$\Phi(f) = \left(\frac{p}{q}, \frac{p'}{q'}\right).$$

Here $\mathbb{Q}^*$ denotes the multiplicative group of the positive rational numbers.

The map is obviously well-defined due to the lemma above, and it is very easy to see that it is a homomorphism of groups. The two components of the map capture the behavior at both ends, eventually near $-\infty$ and eventually near $+\infty$. The two numbers $p/q$ and $p'/q'$ measure the "rate of growth" of the map at both ends.

A corollary of this result is that, as expected, $\mathrm{Com}(F)$ is infinitely generated.

# 5   Commensurations as Quasi-isometries

Let $G$ be a finitely generated group. Quasi-isometries of $G$ can be naturally composed, and there is a natural notion of equivalence class of quasi-isometries. Two quasi-isometries are considered equivalent if they are a bounded distance apart in the sense that $f$ and $g$ are considered equivalent if there exists a number $M > 0$ such that $d(f(t), g(t)) \le M$ for all $t$ in $G$.

Equivalence classes of quasi-isometries form elements of the group of quasi-isometries $QI(G)$ of $G$. It is well known that the commensurator group admits a map to the quasi-isometry group, since all commensurations give maps between finite index subgroups which are canonically quasi-isometric to the ambient group. The result we want to prove in this section is that for Thompson's group $F$, this map is one-to-one.

**Theorem 5.1** *The natural homomorphism* $\mathrm{Com}(F) \to QI(F)$ *is injective.*

We begin with an elementary lemma.

**Lemma 5.2** *Given an element* $\tau \in P$ *which is different from the identity, there exist two intervals* $I$ *and* $J$ *of the real line, whose endpoints are dyadic integers, with* $\tau(I) = J$, *and such that* $I \cap J = \varnothing$.

**Proof** The case when the slope of $\tau$ is always 1 or $-1$ is trivial. For a map $t \mapsto t + k$ has a small interval (of length less than $k$) whose image is disjoint from it. If $\tau = -Id$ the result is trivial.

If the slope is not constantly equal to 1, it has a piece with slope $\pm 2^i$ with $i \neq 0$. Assume without loss of generality (by possibly taking $\tau^{-1}$ instead of $\tau$) that $i > 0$. Hence there are two intervals $[a, b]$ and $[c, d]$ such that $\tau(a) = c$ and $\tau(b) = d$ and also $d - c = 2^i(b - a)$. It is possible that $[a, b]$ and $[c, d]$ overlap, but since $[c, d]$ is much larger than $[a, b]$ (at least twice the size), we can choose as $J$ a small interval inside $[c, d]$ which is disjoint from $[a, b]$. By construction, the preimage $I$ of $J$ is in $[a, b]$, and hence $I$ and $J$ are disjoint. $\qquad\square$

**Proof of Theorem 5.1** We now take a nontrivial $\tau \in \mathrm{Com}(F)$. By the previous lemma, there exist intervals $I$ and $J$ satisfying the conditions stated above and, in addition, that $I$, and hence $J$, have endpoints of the form $k/2^j$ and $(k + 1)/2^j$. We consider all elements of $F$ whose support (that is, the part where they are not the identity) is contained in $I$. Those elements form a subgroup which is isomorphic to $F$ itself. Let $f$ be one such element. Since its support is inside $I$, its image under the commensuration $\tau$, that is, $f^\tau = \tau \circ f \circ \tau^{-1}$, has support inside $J$.

Hence, the distance (inside $F$) from $f$ to $f^\tau$ is given by the distance from the identity to the element $f^\tau f^{-1}$. But this element has its support inside the disjoint union $I \cup J$, and the two parts are independent from each other (one given by $f$ and the other one by $f^\tau$). By work of Cleary and Taback [5], this subgroup—elements with support in $I \cup J$ which is a direct product of two clone subgroups in their terminology—is

quasi-isometrically embedded in $F$. Hence, we can take elements $f_n$ with support inside $I$ with arbitrarily large norm, and hence $f_n^\tau f_n^{-1}$ has also arbitrarily large norm. This proves that the image of $\tau$, a quasi-isometry, is not at bounded distance from the identity and the proof is complete. □

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