A Parameter-Uniform Schwarz Method for a Coupled System of Reaction Diffusion Equations

Meghan Stephens* and Niall Madden†
Department of Mathematics, National University of Ireland, Galway

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Abstract

We consider an arbitrarily sized coupled system of one-dimensional reaction-diffusion problems that are singularly perturbed in nature. We describe an algorithm that uses a discrete Schwarz method on three overlapping subdomains, extending the method in [8] to a coupled system. On each subdomain we use a standard finite difference operator on a uniform mesh. We prove that when appropriate subdomains are used the method produces ε-uniform results. Furthermore we improve upon the analysis of [8] to show that, for small ε, just one iteration is required to achieve the expected accuracy.

Keywords: singularly perturbed; coupled system; domain decomposition.

1 Introduction

We consider the following system of m coupled reaction-diffusion equations: Find $\boldsymbol{u} \in [C^4(0,1)]^m$ such that

$$\boldsymbol{L}\boldsymbol{u} := -\varepsilon^2 \boldsymbol{u}'' + A\boldsymbol{u} = \boldsymbol{f} \qquad \text{in} \quad (0,1), \tag{1}$$

subject to the boundary conditions

$$\boldsymbol{u}(0) = \boldsymbol{b_0}, \qquad \boldsymbol{u}(1) = \boldsymbol{b_1}, \tag{2}$$

where $0 < \varepsilon \ll 1$. The matrix A is assumed to be diagonally dominant and satisfies

$$a_{ij} \begin{cases} > 0 & \text{if} \quad i = j, \\ \leq 0 & \text{if} \quad i \neq j, \end{cases} \tag{3}$$

and for all i

$$\sum_{j=1}^{m} a_{ij} > \alpha^2 > 0. \tag{4}$$

The solutions to problems of this type often exhibit layers in which they change rapidly, causing classical techniques to fail. Our aim is to produce a parameter uniform method where the error is independent of the *singular perturbation parameter*, ε.

Some well known and effective methods of finding numerical solutions to singularly perturbed problems involve using a finite difference method on specially adapted meshes [15, 11, 2]. Shishkin first looked at systems of such equations in [17].

*`Meghan.Stephens@NUIGalway.ie`. Supported by the Irish Research Council for Science, Engineering and Technology.
†`Niall.Madden@NUIGalway.ie`

Finite difference methods on a piecewise uniform *Shishkin* mesh for systems of two equations, each with a different perturbation parameter, are studied in [10, 9, 6]. Madden and Stynes [9] show that the method is at least first order accurate, with Linß and Madden [6] improving on the result to show almost second order convergence. In [7], the analysis is extended to a system of m reaction diffusion equations. They analyse a finite difference method on an arbitrary mesh and the results for Shishkin, Bakhvalov and equidistributed meshes are compared.

A system of m two dimensional reaction-diffusion equations with one perturbation parameter is investigated in [5, 4]. In [5], Kellogg et al. use a piecewise uniform mesh and prove that their method has almost second order convergence. In [4], the system studied does not satisfy a maximum principle, and so a different method of analysis is used to prove parameter uniform convergence for both Shishkin and Bakhvalov meshes. Gracia and Lisbona consider a system of two parabolic reaction-diffusion problems in [3]. They prove that their scheme exhibits almost second order convergence in space and first order convergence in time.

This paper considers a Schwarz domain decomposition method using a finite difference method on overlapping subdomains. The earliest example of a domain decomposition method dates from 1869 [16], and was designed to extend known results for differential equations from regular domains to more complex domains. For a general description of domain decomposition methods see [14] and references therein. These algorithms are especially attractive if one considers the efficiency gained when the algorithm is parallelized.

Our domain decomposition method splits the domain into three subdomains, two of which can be solved simulataniously. This has advantages when the algorithm is applied on a dual processor computer, and when extending the method to two dimensional problems.

Domain decomposition methods for singularly perturbed problems are discussed in [1, 8]. In [11, Chapter 10] a Schwarz method for a one dimensional convection-diffusion problem is described. Of primary interest for this study is the discrete Schwarz method used by MacMullen et al. [8] to approximate a single one dimensional reaction-diffusion equation. Letting u be the solution to the differential equation and $U^{[k]}$ the numerical solution obtained after k iterations of the Schwarz technique, they show that the maximum pointwise error satisfies

$$\|u - U^{[k]}\| \leq C(N^{-1}\ln N)^2 + C2^{-k}.$$

This means that the method is ε-uniform, and that at each iteration the error associated with the domain decomposition method is halved.

However, as we show by our numerical results in Section 3, when ε is small one observes far faster convergence of the iterative scheme. So the goal of this paper is two-fold: to extend the Schwarz method to a system of reaction-diffusion equations, and furthermore to prove that for small ε only one iteration is sufficient.

This structure of this paper is as follows. The algorithm is outlined in Section 2, and in Section 3 numerical results are presented. These motivate the numerical analysis which follows in Section 4.

Notation

We denote C, with or without a subscript, to be a constant independent of ε, N and k. Similarly

$$\boldsymbol{C} = (C, C, C, \dots, C)^T,$$

is a vector of identical constants with the same independencies.

For a domain $\Omega = (a, b)$ we denote $\overline{\Omega} = [a, b]$. Similarly for a mesh we denote $\overline{\Omega}^N = \{a = x_0 < x_1 < \cdots < x_N = b\}$ and $\Omega^N = \{x_1 < \cdots < x_{N-1}\}$.

For a vector $\boldsymbol{y} = (y_0, y_1, \dots, y_m)^T$ we define

$$\|\boldsymbol{y}\| = \max_{p=1,\dots,m} |y_p|.$$

For a real-valued function $y \in C(\Omega)$, we use the norm

$$\|y\|_\Omega = \max_{x\in\Omega} |y(x)|,$$

and the semi-norm

$$|y|_{\Omega,j} = \|y^{(j)}\|_\Omega, \quad j = 0, 1, \dots$$

For a vector valued function $\boldsymbol{z} = (z_0, z_1, \dots, z_m)^T$ define

$$\|\boldsymbol{z}\|_\Omega = \max\{\|z_0\|_\Omega, \|z_1\|_\Omega, \dots, \|z_m\|_\Omega\},$$

and the semi-norm

$$|\boldsymbol{z}|_{\Omega,j} = \max\{|z_0|_{\Omega,j}, |z_1|_{\Omega,j}, \dots, |z_m|_{\Omega,j}\}, \quad j = 0, 1, \dots$$

Given two vector valued functions, $\boldsymbol{z}$ and $\boldsymbol{y}$

$$\boldsymbol{z} \le \boldsymbol{y} \quad \text{if} \quad z_p \le y_p \quad \text{for all} \quad p = 0, 1, \dots m.$$

For a vector of mesh functions $\boldsymbol{Z}(x_i) = (Z_0(x_i), Z_1(x_i), \dots Z_m(x_i))^T$ define

$$\|Z\|_{\Omega^N} = \max_j \left(\max_{x_i\in\Omega^N} |Z_j(x_i)| \right).$$

Let $\overline{Z}$ denote the piecewise linear interpolant of Z.

2 Algorithm

The algorithm is as follows. The domain $\Omega = (0, 1)$ is split into three overlapping subdomains

$$\Omega_L = (0, 2\tau), \qquad \Omega_C = (\tau, 1-\tau), \qquad \Omega_R = (1-2\tau, 1),$$

where we choose the *Shishkin transition point* as in [12],

$$\tau = \min\left\{ \frac{1}{4}, 2\frac{\varepsilon \ln N}{\alpha} \right\}$$

On each subdomain, $\Omega_d = (a, b)$, $d = \{L, C, R\}$, construct a uniform mesh $\overline{\Omega}_d^N : \{a = x_0 < x_1 < ... < x_N = b\}$, with $h_d = x_i - x_{i-1} = (b-a)/N$. Then for each Ω_d^N the discretization is

$$\boldsymbol{L^N U_d}(x_i) := -\varepsilon^2\delta^2 \boldsymbol{U_d}(x_i) + A(x_i)\boldsymbol{U_d}(x_i) = \boldsymbol{f}(x_i), \qquad i = 1, \dots, N-1, \tag{5}$$

where

$$\delta^2 z_d(x_i) := \frac{1}{h_d^2} \left(z_d(x_{i-1}) - 2z_d(x_i) + z_d(x_{i+1}) \right).$$

This leads to a linear system of $m(N-1)$ equations. The coefficient matrix is diagonally dominant with a bandwidth of $m+2$. Thus the equations can be easily solved using standard direct techniques. This is in contrast to approach of some authors, for example [10], who solve the system using a block iterative technique. The iterative procedure starts with

$$\boldsymbol{U}^{[0]}(x_0) = \boldsymbol{b_0}, \quad \boldsymbol{U}^{[0]}(x_N) = \boldsymbol{b_1}, \quad \text{and} \quad \boldsymbol{U}^{[0]}(x_i) = 0 \text{ for } \quad 0 < x_i < 1,$$

and for all $k \geq 1$, $\boldsymbol{U}_L^{[k]}, \boldsymbol{U}_C^{[k]}$ and $\boldsymbol{U}_R^{[k]}$ are defined to be the solutions to

$$\begin{aligned}
&\boldsymbol{L^N}\boldsymbol{U}_L^{[k]}(x_i) = \boldsymbol{f}(x_i) \quad x_i \in \Omega_L^N, && \boldsymbol{U}_L^{[k]}(0) = \boldsymbol{b_0}, && \boldsymbol{U}_L^{[k]}(2\tau) = \overline{\boldsymbol{U}}^{[k-1]}(2\tau),\\
&\boldsymbol{L^N}\boldsymbol{U}_R^{[k]}(x_i) = \boldsymbol{f}(x_i) \quad x_i \in \Omega_R^N, && \boldsymbol{U}_R^{[k]}(1-2\tau) = \overline{\boldsymbol{U}}^{[k-1]}(1-2\tau), && \boldsymbol{U}_R^{[k]}(1) = \boldsymbol{b_1},\\
&\boldsymbol{L^N}\boldsymbol{U}_C^{[k]}(x_i) = \boldsymbol{f}(x_i) \quad x_i \in \Omega_C^N, && \boldsymbol{U}_C^{[k]}(\tau) = \overline{\boldsymbol{U}}_L^{[k]}(\tau), && \boldsymbol{U}_C^{[k]}(1-\tau) = \overline{\boldsymbol{U}}_R^{[k]}(1-\tau).
\end{aligned}$$

Then $\boldsymbol{U}^{[k]}$ is taken to be

$$\boldsymbol{U}^{[k]} = \begin{cases} \boldsymbol{U}_L^{[k]}(x_i), & x_i \in \overline{\Omega}_L^N \backslash \overline{\Omega}_C,\\ \boldsymbol{U}_C^{[k]}(x_i), & x_i \in \overline{\Omega}_C^N,\\ \boldsymbol{U}_R^{[k]}(x_i), & x_i \in \overline{\Omega}_R^N \backslash \overline{\Omega}_C. \end{cases}$$

Take $\overline{\Omega}^N = \left(\overline{\Omega}_L^N \backslash \overline{\Omega}_C\right) \bigcup \overline{\Omega}_C^N \bigcup \left(\overline{\Omega}_R^N \backslash \overline{\Omega}_C\right)$. The algorithm terminates when

$$\|\boldsymbol{U}^{[k]} - \boldsymbol{U}^{[k-1]}\|_{\overline{\Omega}^N} \leq \lambda N^{-2}. \tag{6}$$

Here λ is user-chosen parameter selected to ensure that the difference between successive iterates, relative to the magnitude of the true solution, is $\mathcal{O}(N^{-2})$. One should take λ to be $\mathcal{O}(\|\boldsymbol{u}\|_{\overline{\Omega}})$, furthermore this may be estimated *a priori* by noting that, as shown in Lemma 3,

$$\|\boldsymbol{u}\|_{\overline{\Omega}} \leq \max\{\|\boldsymbol{b_0}\|, \|\boldsymbol{b_1}\|\} + \alpha^{-2}\|\boldsymbol{f}\|_{\overline{\Omega}}.$$

3 Numerical Results

In this section we present numerical results which will be substantiated with the theoretical results in the proceeding section. The exact solution to the test problem is unknown so we use a double mesh method to estimate the errors. This estimate is given by

$$D_N := \|\boldsymbol{U}_N - \overline{\boldsymbol{U}}_{\widetilde{2N}}\|_{\Omega^N},$$

where $\boldsymbol{U}_N$ is the result of the algorithm with N discretization intervals in each subdomain and $\boldsymbol{U}_{\widetilde{2N}}$ is the numerical solution obtained on on a mesh with the same transition points, but $2N$ intervals in each subdomain. See [11] for a mathematical justification of the double mesh technique. We calculate the numerical rates of convergence using

$$\rho_N := \log_2\left(\frac{D_N}{D_{2N}}\right).$$

Our test problem is a system of four equations with

$$A = \begin{pmatrix} 2(x+1)^2 & -(1+x^3) & -0.1 & -0.2\\ -2\cos\frac{\pi x}{4} & (1+\sqrt{2})e^{1-x} & -0.2 & -0.1\\ -2\cos\frac{\pi x}{4} & -\frac{1}{2}(x+1)^2 & 2(1+\sqrt{2})e^{1-x} & -\cos\frac{\pi}{5}\\ -(1+x^3) & -0.1 & -0.2 & 3(x+1)^3 \end{pmatrix},$$

and

$$\boldsymbol{b_0} = \begin{pmatrix} 0\\0\\1\\2 \end{pmatrix}, \qquad \boldsymbol{b_1} = \begin{pmatrix} 0\\0\\1\\2 \end{pmatrix}, \qquad \boldsymbol{f} = \begin{pmatrix} 2+x\\1\\2e^x\\0.1 \end{pmatrix}.$$

For these experiments, the user-chosen parameter in (6) is taken to be $\lambda = 2$.

$N =$	64		128		256		512		1024		2048		
ε	D_N	k	D_N	k	D_N	k	D_N	k	D_N	k	D_N	k	
$\mathbf{2^{0}}$	1.73e-04	7	2.00e-05	9	1.11e-05	10	1.28e-06	12	7.11e-07	13	8.20e-08	15	
	3.11		0.85		3.12		0.85		3.12		0.85		$\rho^{\mathbf{N}}$
$\mathbf{2^{-1}}$	1.50e-04	5	3.78e-05	6	9.48e-06	7	2.38e-06	8	5.97e-07	9	1.50e-07	10	
	1.99		1.99		1.99		2.00		2.00		2.00		$\rho^{\mathbf{N}}$
$\mathbf{2^{-2}}$	5.70e-04	3	1.43e-04	4	3.58e-05	4	8.96e-06	5	2.24e-06	5	5.60e-07	5	
	1.99		2.00		2.00		2.00		2.00		2.00		$\rho^{\mathbf{N}}$
$\mathbf{2^{-3}}$	2.16e-03	2	5.45e-04	2	1.37e-04	2	3.42e-05	3	8.56e-06	3	2.14e-06	3	
	1.99		1.99		2.00		2.00		2.00		2.00		$\rho^{\mathbf{N}}$
$\mathbf{2^{-4}}$	8.03e-03	2	2.11e-03	2	5.32e-04	2	1.34e-04	2	3.34e-05	2	8.36e-06	2	
	1.93		1.99		1.99		2.00		2.00		2.00		$\rho^{\mathbf{N}}$
$\mathbf{2^{-5}}$	2.85e-02	1	7.94e-03	1	2.09e-03	1	5.26e-04	1	1.32e-04	1	3.31e-05	1	
	1.84		1.93		1.99		1.99		2.00		2.00		$\rho^{\mathbf{N}}$
$\mathbf{2^{-6}}$	3.64e-02	1	1.49e-02	1	5.20e-03	1	1.67e-03	1	5.19e-04	1	1.31e-04	1	
	1.29		1.52		1.64		1.68		1.98		2.00		$\rho^{\mathbf{N}}$
$\mathbf{2^{-7}}$	3.63e-02	1	1.49e-02	1	5.18e-03	1	1.66e-03	1	5.17e-04	1	1.57e-04	1	
	1.29		1.52		1.64		1.68		1.72		1.75		$\rho^{\mathbf{N}}$
$\mathbf{2^{-8}}$	3.63e-02	1	1.48e-02	1	5.17e-03	1	1.66e-03	1	5.17e-04	1	1.57e-04	1	
	1.29		1.52		1.64		1.68		1.72		1.75		$\rho^{\mathbf{N}}$
$\mathbf{2^{-9}}$	3.63e-02	1	1.48e-02	1	5.17e-03	1	1.66e-03	1	5.16e-04	1	1.57e-04	1	
	1.29		1.52		1.64		1.68		1.72		1.75		$\rho^{\mathbf{N}}$
$\mathbf{2^{-10}}$	3.62e-02	1	1.48e-02	1	5.17e-03	1	1.66e-03	1	5.16e-04	1	1.57e-04	1	
	1.29		1.52		1.64		1.68		1.72		1.75		$\rho^{\mathbf{N}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\mathbf{2^{-20}}$	3.62e-02	1	1.48e-02	1	5.17e-03	1	1.66e-03	1	5.16e-04	1	1.57e-04	1	
	1.29		1.52		1.64		1.68		1.72		1.75		$\rho^{\mathbf{N}}$

Table 1: Numerical results for the test problem.

Table 1 above lists D_N, ρ_N and k (the number of iterations computed) for various values of N and ε. We can see that the errors are independent of the singular perturbation parameter ε and are decreasing as N increases. The computed rates of convergence are second order, with the usual $\log N$ factor associated with these techniques. For large ε the number of iterations increase slightly with N, however for small ε only one iteration of the Schwarz method was required.

To ascertain the effect of the choice of the parameter λ in (6) on the accuracy and number of iterations required, we conducted further numerical experiments with $\lambda = 0.1$ and 10. We found that, when ε was large, taking $\lambda = 0.1$ resulted in a small increase in the number of iterations required and a small reduction in the error. For $\lambda = 10$ and ε large fewer iterations are needed and the errors are larger. However, when $\varepsilon \leq 2^{-5}$, the results are identical.

4 Numerical Analysis of the Schwarz algorithm

Introduction

In [8] it is shown that, for a single reaction-diffusion problem

$$\|u - U^{[k]}\|_{\overline{\Omega}^N} \leq C(N^{-1} \ln N)^2 + C2^{-k}.$$

However the numerical results suggest that the algorithm converges at a much faster rate. In this paper we prove that, when ε is small,

$$\|\boldsymbol{u} - \boldsymbol{U}^{[k]}\|_{\overline{\Omega}^N} \leq C_0(N^{-1} \ln N)^2 + C_1 N^{-2k}.$$

meaning that we obtain the desired accuracy after only one iteration.

The analysis proceeds as follows. In Lemma 6 we prove a result that is a similar to [8, Theorem 1], but for a coupled system of m reaction-diffusion problems. In Lemma 7 we show how the solution to the discrete problem may be bounded by the solution to an uncoupled, constant coefficient problem.

This is then used in Lemma 8 to show that, for small ε, successive Schwarz iterates differ by less than CN^{-2}. We combine these results to prove the main result of this paper in Theorem 4.

The analysis of reaction-diffusion problems often involves a *Shishkin Decomposition*, see for example [11, 2], which splits the solution into smooth and singular components. Using now standard arguments such as those in [10], [9] and [7] we can prove the following lemma.

Lemma 1. *The solution to* (1) *can be decomposed into*

$$\boldsymbol{u}(x) = \boldsymbol{v}(x) + \boldsymbol{w}(x),$$

with

$$|\boldsymbol{v}|_{\overline{\Omega},j} \leq C(1+\varepsilon^{2-j}), \qquad \text{for } j = 0, 1, \ldots, 4, \tag{7}$$

and for $x \in \overline{\Omega}$

$$\|\boldsymbol{w}^{(\boldsymbol{j})}(x)\| \leq C\varepsilon^{-j}(e^{-x\alpha/\varepsilon} + e^{-(1-x)\alpha/\varepsilon}), \qquad \text{for } j = 0, 1, \ldots, 4. \tag{8}$$

Our analysis makes extensive use of the following maximum principle (see [13] for details of maximum principles).

Lemma 2. *Maximum Principle: Let* $\boldsymbol{z}$ *be a continuous vector function defined in the domain* $\overline{\Omega} = [a, b]$ *with* $\boldsymbol{L}\boldsymbol{z} \geq \boldsymbol{0}$ *on* $\Omega = (a, b)$, $\boldsymbol{z}(a) \geq \boldsymbol{0}$ *and* $\boldsymbol{z}(b) \geq \boldsymbol{0}$. *Then* $\boldsymbol{z} \geq \boldsymbol{0}$ *in* Ω.

Proof. The proof is similar to that given for partial differential equations in [5]. It should be noted that the assumptions given in (3)–(4) are needed. □

Lemma 3. *If* $\boldsymbol{L}\boldsymbol{z}(x) = \boldsymbol{g}(x)$ *for all* $x \in \Omega = (a, b)$, $\boldsymbol{z}(a) = \boldsymbol{z_a}$ *and* $\boldsymbol{z}(b) = \boldsymbol{z_b}$ *then*

$$\|\boldsymbol{z}\|_{\overline{\Omega}} \leq \max\{\|\boldsymbol{z_a}\|, \|\boldsymbol{z_b}\|\} + \frac{\|\boldsymbol{g}\|_{\overline{\Omega}}}{\alpha^2}.$$

Proof. The proof is a direct consequence of Lemma 2. □

The following are analogous results for the discrete problem.

Lemma 4. *(Discrete Maximum Principle) Let* $\boldsymbol{L^N}$ *be defined as in* (5) *and let* $\boldsymbol{Z}(x_i)$ *be a mesh function defined on* $\overline{\Omega}^N := \{x_0 < x_1 < \ldots < x_N\}$. *If* $\boldsymbol{L^N}\boldsymbol{Z}(x_i) \geq \boldsymbol{0}$ *for all* $x_i \in \Omega^N$, $\boldsymbol{Z}(x_0) \geq \boldsymbol{0}$ *and* $\boldsymbol{Z}(x_N) \geq \boldsymbol{0}$. *Then* $\boldsymbol{Z}(x_i) \geq \boldsymbol{0}$ *for all* $x_i \in \overline{\Omega}^N$.

Proof. The matrix associated with $\boldsymbol{L^N}$ is an M-Matrix. □

Lemma 5. *Let* $\boldsymbol{Z}(x_i)$ *be a mesh function defined on* $\overline{\Omega}^N := \{x_0 < x_1 < \ldots < x_N\}$. *If* $L^N\boldsymbol{Z}(x_i) = \boldsymbol{g}(x_i)$ *for all* $x_i \in \Omega^N$, $\boldsymbol{Z}(x_0) = \boldsymbol{b}_0$ *and* $\boldsymbol{Z}(x_N) = \boldsymbol{b}_1$ *then*

$$\|\boldsymbol{Z}\|_{\overline{\Omega}^N} \leq \max\{\|\boldsymbol{b}_0\|, \|\boldsymbol{b}_1\|\} + \frac{\|\boldsymbol{g}\|_{\overline{\Omega}^N}}{\alpha^2}.$$

Proof. The proof is a direct consequence of the Discrete Maximum Principle, Lemma 4. □

The next Lemma establishes that the subdomain iterations converge independently of ε and that the discretization error is parameter uniform.

Lemma 6. *Let* $\boldsymbol{u}$ *be the solution to (1)–(2) and let* $\boldsymbol{U}^{[k]}$ *be the* k^{th} *iterate of the discrete Schwarz method described in Section 2. Then, there are constants* C_0 *and* C_1 *such that*

$$\|\boldsymbol{U}^{[k]} - \boldsymbol{u}\|_{\Omega^N} \leq C_0 2^{-k} + C_1(N^{-1}\ln N)^2.$$

Proof. At the first iteration $(\boldsymbol{U}^{[0]} - \boldsymbol{u})(0) = 0$ and $(\boldsymbol{U}^{[0]} - \boldsymbol{u})(1) = 0$. Also because $\boldsymbol{U}^{[0]}(x_i) = \boldsymbol{0}$ for $x_i \in \Omega^N := \{x_1 < x_2 < ... < x_{N-1}\}$, we can use Lemma 3 to show that

$$\|\boldsymbol{U}^{[0]} - \boldsymbol{u}\|_{\Omega^N} = \|\boldsymbol{u}\|_{\Omega^N} \leq C.$$

Clearly there exists C_0 and C_1 such that

$$\|\boldsymbol{U}^{[0]} - \boldsymbol{u}\|_{\overline{\Omega}^N} \leq C_0 2^0 + C_1(N^{-1}\ln N)^2.$$

Assume that for an arbitrary integer $k \geq 0$ there exists C_0 and C_1 such that

$$\|\boldsymbol{U}^{[k]} - \boldsymbol{u}\|_{\overline{\Omega}^N} \leq C_0 2^{-k} + C_1(N^{-1}\ln N)^2.$$

A standard truncation error estimate for $z \in C^4(x_{i-1}, x_i)$ on a uniform mesh is

$$\left\| \left(\delta^2 - \frac{d^2}{dx^2}\right) z(x_i) \right\|_{(x_{i-1},x_i)} \leq \frac{(x_i - x_{i-1})^2}{12} |z|_{(x_{i-1},x_i),4}. \tag{9}$$

On $\overline{\Omega}_L^N$, note that $|\boldsymbol{u}|_{\Omega_L^N,4} \leq C\varepsilon^{-4}$ and $h_L \leq 4(\varepsilon \ln N)/(\alpha N)$. Now

$$\begin{aligned}
\|L^N(\boldsymbol{U}_L^{[k+1]} - \boldsymbol{u})\|_{\Omega_L^N} &= \|\boldsymbol{f} - L^N \boldsymbol{u}\|_{\Omega_L^N} \\
&= \|(L - L^N)\boldsymbol{u}\|_{\Omega_L^N} \\
&= \left\| \varepsilon^2 \left(\delta^2 - \frac{d^2}{dx^2}\right) \boldsymbol{u} \right\|_{\Omega_L^N} \\
&\leq \varepsilon^2 \frac{h_L^2}{12} |\boldsymbol{u}|_{\Omega_L^N,4} \\
&\leq \frac{\varepsilon^2}{12} \left(\frac{4\varepsilon \ln N}{\alpha N}\right)^2 C_2 \varepsilon^{-4} \\
&\leq C(N^{-1}\ln N)^2,
\end{aligned}$$

for some C. The end point of the subdomain Ω_L^N is 2τ, which in general is not in Ω^N, so we use a piecewise linear interpolant of the previous iterate to determine $\boldsymbol{U}_L^{[k+1]}(2\tau)$. In order to put a bound on $\|(\overline{\boldsymbol{u}} - \boldsymbol{u})(2\tau)\|$ at we must decompose u as in Lemma 1 to give us

$$\|(\overline{\boldsymbol{u}} - \boldsymbol{u})(2\tau)\| \leq \|(\overline{\boldsymbol{v}} - \boldsymbol{v})(2\tau)\| + \|(\overline{\boldsymbol{w}} - \boldsymbol{w})(2\tau)\|$$

Let $\overline{z}$ be the piecewise linear interpolant to $z \in C^2(x_{i-1}, x_i)$, then standard error estimates give

$$\|z - \overline{z}\|_{(x_{i-1},x_i)} \leq C(x_i - x_{i-1})^2 |z|_{(x_{i-1},x_i),2}. \tag{10}$$

Thus

$$\|(\overline{\boldsymbol{v}} - \boldsymbol{v})(2\tau)\| \leq C h_c^2 \|v''(2\tau)\| \leq C_3 N^{-2}.$$

For the interpolant of the singular component first consider $\tau = 1/4$. Then $\varepsilon^{-1} \leq 8 \ln N/\alpha$ and

$$\|(\overline{\boldsymbol{w}} - \boldsymbol{w})(2\tau)\| \leq C h_c^2 \|w''(2\tau)\| \leq C\varepsilon^{-2} N^{-2} \leq C_3 (N^{-1}\ln N)^2.$$

For $\tau = 2\varepsilon \ln N/\alpha$ note that the layer function $\boldsymbol{w}$ is monotonic in the regions $(\tau, 1/2)$ and $(1/2, 1-\tau)$. Hence

$$\|(\overline{\boldsymbol{w}} - \boldsymbol{w})(2\tau)\| \leq C\|\boldsymbol{w}\|_{\Omega_C} \leq C 2 e^{\frac{-\tau\alpha}{\varepsilon}} \leq C_5 N^{-2}.$$

Now, using our inductive argument and these bounds,

$$\begin{aligned}
\|(\boldsymbol{U}_L^{[k+1]} - \boldsymbol{u})(2\tau)\| &= \|(\overline{\boldsymbol{U}}^{[k]} - \boldsymbol{u})(2\tau)\| \\
&\le \|(\overline{\boldsymbol{U}}^{[k]} - \overline{\boldsymbol{u}})(2\tau)\| + \|(\overline{\boldsymbol{u}} - \boldsymbol{u})(2\tau)\| \\
&\le C_6 2^{-k} + C_7(N^{-1}\ln N)^2 + C_8(N^{-1}\ln N)^2 \\
&\le C_6 2^{-k} + C_9(N^{-1}\ln N)^2.
\end{aligned}$$

Consider the mesh functions

$$\boldsymbol{\Psi}(x_i) = \frac{x_i}{2\tau}\boldsymbol{C_0}2^{-k} + \boldsymbol{C_1}(N^{-1}\ln N)^2 \pm (\boldsymbol{U}_L^{[k+1]} - \boldsymbol{u})(x_i)$$

where C_1 is taken to be $\max\{C_3/\alpha^2, C_9\}$ and $C_0 \ge C_6$. Then, for $x_i \in \Omega_L^N$

$$L^N\boldsymbol{\Psi}(x_i) \ge A(x_i)\frac{x_i}{2\tau}\boldsymbol{C_0}2^{-k} + A(x_i)\boldsymbol{C_1}(N^{-1}\ln N)^2 - \boldsymbol{C_3}(N^{-1}\ln N)^2 \ge \boldsymbol{0},$$

$$\boldsymbol{\Psi}(0) = \boldsymbol{C_1}(N^{-1}\ln N)^2 \pm 0 \ge 0,$$

and

$$\boldsymbol{\Psi}(2\tau) \ge \boldsymbol{C_0}2^{-k} + \boldsymbol{C_1}(N^{-1}\ln N)^2 - \left(\boldsymbol{C_6}2^{-k} + \boldsymbol{C_9}(N^{-1}\ln N)^2\right) \ge \boldsymbol{0}.$$

Now, using the Discrete Maximum Principle (Lemma 4), $\boldsymbol{\Psi}(x_i) \ge \boldsymbol{0}$, that is

$$\frac{x_i}{2\tau}\boldsymbol{C_0}2^{-k} + \boldsymbol{C_1}(N^{-1}\ln N)^2 \ge |(\boldsymbol{U}_L^{[k+1]} - \boldsymbol{u})(x_i)|,$$

and consequently

$$C_0 2^{-(k+1)} + C_1(N^{-1}\ln N)^2 \ge \|\boldsymbol{U}_L^{[k+1]} - \boldsymbol{u}\|_{\overline{\Omega}_L^N \setminus \overline{\Omega}_C}.$$

A similar argument can be used to find C_0 and C_1 such that

$$C_0 2^{-(k+1)} + C_1(N^{-1}\ln N)^2 \ge \|\boldsymbol{U}_R^{[k+1]} - \boldsymbol{u}\|_{\overline{\Omega}_L^N \setminus \overline{\Omega}_C}.$$

In order to use the same technique in $\overline{\Omega}_C^N$ we must be able to put a bound on $\|(L^N - L)u\|_{\Omega_C^N}$. Using (9),

$$\begin{aligned}
\|(L^N - L)\boldsymbol{v}\|_{\Omega_C^N} &\le \varepsilon^2\frac{h_C^2}{12}|\boldsymbol{v}|_{\Omega_C,4} \\
&\le \frac{\varepsilon^2}{12}\left(\frac{1-2\tau}{N}\right)^2 C(1+\varepsilon^{-2}) \\
&\le C_1 N^{-2}.
\end{aligned}$$

Suppose first that $\tau = 1/4$, and so $\varepsilon^{-1} \le 8\log N/\alpha$. Then

$$\begin{aligned}
\|(L^N - L)\boldsymbol{w}\|_{\Omega_C^N} &\le \varepsilon^2\frac{h_C^2}{12}|\boldsymbol{w}|_{\Omega_C,4} \\
&\le C\frac{\varepsilon^2}{12}N^{-2}\varepsilon^{-4} \\
&\le CN^{-2}\left(\frac{8\ln N}{\alpha}\right)^2 \\
&\le C_2\left(N^{-1}\ln N\right)^2.
\end{aligned}$$

Otherwise, suppose that $\tau = 2\varepsilon \ln N/\alpha$. A standard Taylor expansion gives that for any $z \in C^2(x_{i-1}, x_i)$

$$|\delta^2 z(x_i)| \le |z''(x)|_{(x_{i-1},x_i)}.$$

Hence,

$$\begin{aligned}
\|(L^N - L)\boldsymbol{w}\|_{\Omega_C^N} &= \left\|\varepsilon^2\left(\delta^2 - \frac{d^2}{dx^2}\right)\boldsymbol{w}\right\|_{\Omega_C} \\
&\le 2\varepsilon^2 |\boldsymbol{w}|_{\Omega_C,2} \\
&\le 2C(e^{-x_i\alpha/\varepsilon} + e^{-(1-x_i)\alpha/\varepsilon}), \qquad \text{for } x_i \in \Omega_C^N.
\end{aligned}$$

If $\tau < x_i \le 1/2$,

$$e^{-x_i\alpha/\varepsilon} + e^{-(1-x_i)\alpha/\varepsilon} \le 2e^{-x_i\alpha/\varepsilon} < 2e^{-\tau\alpha/\varepsilon} = 2e^{-2\ln N} = 2N^{-2}.$$

The analogous result holds for $1/2 \le x_i < 1 - \tau$. This means that

$$\|(L^N - L)\boldsymbol{w})\|_{\Omega_C^N} \le C_2\left(N^{-1}\ln N\right)^2.$$

So using this decomposition we find that

$$\begin{aligned}
\|L^N({\boldsymbol{U_C}}^{[k+1]} - \boldsymbol{u})\|_{\Omega_C^N} &= \|(L^N - L)\boldsymbol{u}\|_{\Omega_C^N} \\
&\le \|(L^N - L)\boldsymbol{v}\|_{\Omega_C^N} + \|(L^N - L)\boldsymbol{w}\|_{\Omega_C^N} \\
&\le C_3\left(N^{-1}\ln N\right)^2.
\end{aligned}$$

Also there exists C_0 and C_5 such that

$$\|({\boldsymbol{U_C}}^{[k+1]} - \boldsymbol{u})(\tau)\| = \|({\boldsymbol{U_L}}^{[k+1]} - \boldsymbol{u})(\tau)\| \le C_0 2^{-(k+1)} + C_5(N^{-1}\ln N)^2,$$

and

$$\|({\boldsymbol{U_C}}^{[k+1]} - \boldsymbol{u})(1-\tau)\| = \|({\boldsymbol{U_R}}^{[k+1]} - \boldsymbol{u})(1-\tau)\| \le C_0 2^{-(k+1)} + C_5(N^{-1}\ln N)^2,$$

so using Lemma 5

$$\begin{aligned}
\|({\boldsymbol{U_C}}^{[k+1]} - \boldsymbol{u})\|_{\Omega_C^N} &\le \frac{C_3}{\alpha^2}(N^{-1}\ln N)^2 + C_0 2^{-(k+1)} + C_5(N^{-1}\ln N)^2 \\
&\le C_0 2^{-(k+1)} + C_1(N^{-1}\ln N)^2.
\end{aligned}$$

Consequently

$$\|({\boldsymbol{U_C}}^{[k+1]} - \boldsymbol{u})\|_{\Omega^N} \le C_0 2^{-(k+1)} + C_1(N^{-1}\ln N)^2.$$

$\square$

From Lemma 6 one can deduce that $2\log_2(N/\ln N)$ iterations of the Schwarz scheme are required to ensure that $\|\boldsymbol{u} - \boldsymbol{U}^{[k]}\| \le C(N^{-1}\ln N)^2$. In fact, this is almost exactly as found in Table 1 for $\varepsilon = 1$. However, that table also shows that, for small ε, only one iteration is required. The remainer of this paper is concerned with proving that is the case. Therefore, from this point we are primarily concerned with the case where $2\varepsilon \ln N/\alpha < 1/4$.

Our next lemma shows that solutions to the discrete problem can be bounded by solutions to a problem with a constant coefficients.

Lemma 7. *Suppose that $\boldsymbol{Z}(x_i)$ is the solution to*

$$L^N \boldsymbol{Z}(x_i) := -\varepsilon^2\delta^2\boldsymbol{Z}(x_i) + A(x_i)\boldsymbol{Z}(x_i) = 0 \quad i = 1, \dots N-1, \qquad \boldsymbol{Z}(x_0) = \boldsymbol{z}_0, \qquad \boldsymbol{Z}(x_N) = \boldsymbol{z}_1,$$

and $\boldsymbol{Y}(x_i)$ solves the uncoupled problem

$$-\varepsilon^2\delta^2\boldsymbol{Y}(x_i) + \alpha^2\boldsymbol{Y}(x_i) = 0 \quad i = 1, \dots N-1, \qquad \boldsymbol{Y}(x_0) = \boldsymbol{y_0}, \qquad \boldsymbol{Y}(x_N) = \boldsymbol{y_1}, \tag{11}$$

where $y_0 = \|\boldsymbol{z}_0\|$ and $y_1 = \|\boldsymbol{z}_1\|$. Then

$$\boldsymbol{Y}(x_i) \geq \boldsymbol{Z}(x_i).$$

Proof. Note that $\boldsymbol{Y} \geq \boldsymbol{0}$ and furthermore, since all the equations in (11) are identical, $\boldsymbol{Y} = \{Y, Y, \dots, Y\}^T$. Consequently, $A(x_i)\boldsymbol{Y}(x_i) \geq \alpha^2\boldsymbol{Y}(x_i)$ for $i = 1, 2 \dots N-1$, and thus

$$\begin{aligned} L^N(\boldsymbol{Y}(x_i) - \boldsymbol{Z}(x_i)) &= -\varepsilon^2\delta^2\boldsymbol{Y}(x_i) + A(x_i)\boldsymbol{Y}(x_i) + L^N\boldsymbol{Z}(x_i) \\ &\geq -\varepsilon^2\delta^2\boldsymbol{Y}(x_i) + \alpha^2\boldsymbol{Y}(x_i) + 0 \\ &= 0. \end{aligned}$$

Thus, as $\boldsymbol{Y}(x_0) - \boldsymbol{Z}(x_0) \geq \boldsymbol{0}$, $\boldsymbol{Y}(x_N) - \boldsymbol{Z}(x_N) \geq \boldsymbol{0}$, we can use the Discrete Maximum Principle of Lemma 4 to prove that

$$\boldsymbol{Y}(x_i) \geq \boldsymbol{Z}(x_i), \qquad i = 0, 1 \dots N.$$

□

Now we will show that the discrete Schwarz iterates converge at a higher rate than is suggested by Lemma 6.

Lemma 8. *Let $\boldsymbol{U}^{[k]}(x_i)$ be the k^{th} iterate of the discrete Schwarz method described in Section 2. Then there exists some C such that*

$$\|\boldsymbol{U}^{[k+1]} - \boldsymbol{U}^{[k]}\|_{\overline{\Omega}^N} \leq Cp^k \quad \textit{where} \quad p = \left(1 + \frac{\tau\alpha}{\varepsilon N}\right)^{-N} < 1.$$

Furthermore if $\tau = 2\varepsilon \ln N/\alpha$ then $p \leq 4N^{-2}$.

Proof. At the first iteration $\|\boldsymbol{U}^{[0]}\|_{\Omega^N} = 0$ so clearly $\|\boldsymbol{U}^{[1]} - \boldsymbol{U}^{[0]}\|_{\Omega^N} = \|\boldsymbol{U}^{[1]}\|_{\Omega^N}$. Using Lemma 5

$$\|\boldsymbol{U}_L^{[1]}\|_{\overline{\Omega}_L^N} \leq \|\boldsymbol{b}_0\| + \frac{\|f\|_{\overline{\Omega}_L^N}}{\alpha^2} \leq C.$$

Similarly $\|\boldsymbol{U}_R^{[1]}\|_{\overline{\Omega}_R^N} \leq C$. Also $\boldsymbol{U}_C^{[1]}$ satisfies

$$L^N\boldsymbol{U}_C^{[1]}(x_i) = \boldsymbol{f} \quad \text{for } x_i \in \Omega_C^N, \qquad \boldsymbol{U}_C^{[1]}(\tau) = \boldsymbol{U}_L^{[1]}(\tau), \qquad \boldsymbol{U}_C^{[1]}(1-\tau) = \boldsymbol{U}_R^{[1]}(1-\tau),$$

so we can apply Lemma 5 to find that

$$\|\boldsymbol{U}_C^{[1]}\|_{\overline{\Omega}_C^N} \leq C + \frac{\|f\|_{\overline{\Omega}_C^N}}{\alpha^2} \leq C_1.$$

Combining these results we see that

$$\|\boldsymbol{U}^{[1]} - \boldsymbol{U}^{[0]}\|_{\overline{\Omega}^N} \leq Cp^0$$

Assume that for an arbitrary integer $k \geq 0$,

$$\|\boldsymbol{U}^{[k+1]} - \boldsymbol{U}^{[k]}\|_{\overline{\Omega}^N} \leq Cp^k \quad \text{where} \quad p = \left(1 + \frac{\tau\alpha}{\varepsilon N}\right)^{-N}.$$

In Lemma 5.1 of [11] it is shown that, for all integers $N \geq 0$,

$$\left(1 + \frac{2\ln N}{N}\right)^{-\frac{N}{2}} \leq 2N^{-1}.$$

Thus, if $\tau = 2\varepsilon \ln N/\alpha$, then

$$\left(1 + \frac{\tau\alpha}{\varepsilon N}\right)^{-N} = \left(1 + 2\frac{\ln N}{N}\right)^{-N} \leq 4N^{-2}. \tag{12}$$

Now

$$L^N(\boldsymbol{U}_L^{[k+2]} - \boldsymbol{U}_L^{[k+1]})(x_i) = \mathbf{0} \quad \text{for } x_i \in \Omega_L^N, \qquad (\boldsymbol{U}_L^{[k+2]} - \boldsymbol{U}_L^{[k+1]})(0) = \mathbf{0},$$

and using our inductive hypotheses

$$\|(\boldsymbol{U}_L^{[k+2]} - \boldsymbol{U}_L^{[k+1]})(2\tau)\| = \|(\overline{\boldsymbol{U}}^{[k+1]} - \overline{\boldsymbol{U}}^{[k]})(2\tau)\| \leq Cp^k.$$

Let $E_L^{[k+2]}(x_i)$ be the solution to

$$-\varepsilon^2\delta^2 E_L^{[k+2]}(x_i) + \alpha^2 E_L^{[k+2]}(x_i) = 0, \quad x_i \in \Omega_L^N, \qquad E_L^{[k+2]}(0) = 0, \qquad E_L^{[k+2]}(2\tau) = Cp^k.$$

Then using Lemma 7 $(\boldsymbol{U}^{[k+2]} - \boldsymbol{U}^{[k+1]})(x_i) \leq \boldsymbol{E}_L^{[k+2]}(x_i)$. The exact solution to this difference equation is

$$E_L^{[k+2]}(x_i) = Cp^k \frac{(A+B)^i - (A-B)^i}{(A+B)^N - (A-B)^N},$$

where $A = 1 + 2\big(\frac{\tau\alpha}{\varepsilon N}\big)^2$ and $B = 2\frac{\tau\alpha}{\varepsilon N}\sqrt{1 + (2\frac{\tau\alpha}{\varepsilon N})^2}$. This means that for $x_i \in \Omega_L^N \backslash \Omega_C$

$$\begin{aligned}
E_L^{[k+2]}(x_i) &\leq Cp^k \frac{(A+B)^{N/2} - (A-B)^{N/2}}{(A+B)^N - (A-B)^N} \\
&= \frac{Cp^k}{(A+B)^{N/2} + (A-B)^{N/2}} \\
&\leq \frac{Cp^k}{(A+B)^{N/2}} \\
&\leq \frac{Cp^k}{(A + 2\frac{\tau\alpha}{\varepsilon N})^{N/2}} \\
&= \frac{Cp^k}{(1 + \big(\frac{\tau\alpha}{\varepsilon N}\big)^2 + 2\frac{\tau\alpha}{\varepsilon N})^{N/2}} \\
&= Cp^k(1 + \frac{\tau\alpha}{\varepsilon N})^{-N} \\
&= Cp^{k+1}.
\end{aligned}$$

Consequently

$$\|\boldsymbol{U}_L^{[k+2]} - \boldsymbol{U}_L^{[k+1]}\|_{\overline{\Omega}_L^N \backslash \overline{\Omega}_C} \leq Cp^{k+1}. \tag{13}$$

Similar arguments can be used to show that

$$\|\boldsymbol{U}_R^{[k+2]} - \boldsymbol{U}_R^{[k+1]}\|_{\overline{\Omega}_R^N \backslash \overline{\Omega}_C} \leq Cp^{k+1}. \tag{14}$$

Finally we note that $L^N(\boldsymbol{U}_C^{[k+2]} - \boldsymbol{U}_C^{[k+1]})(x_i) = \mathbf{0}$ for all $x_i \in \Omega_C^N$,

$$|(\boldsymbol{U}_C^{[k+2]} - \boldsymbol{U}_C^{[k+1]})(\tau)| = |(\boldsymbol{U}_L^{[k+2]} - \boldsymbol{U}_L^{[k+1]})(\tau)|,$$

$$|(\boldsymbol{U}_C^{[k+2]} - \boldsymbol{U}_C^{[k+1]})(1-\tau)| = |(\boldsymbol{U}_R^{[k+2]} - \boldsymbol{U}_R^{[k+1]})(1-\tau)|.$$

Now using (13) and (14) with Lemma 5

$$\|\boldsymbol{U}_C^{[k+2]} - \boldsymbol{U}_C^{[k+1]}\|_{\overline{\Omega}_C^N} \leq Cp^{k+1}.$$

Combining the results we find that

$$\|\boldsymbol{U}^{[k+2]} - \boldsymbol{U}^{[k+1]}\|_{\overline{\Omega}^N} \leq Cp^{k+1}.$$

$\square$

The following theorem contains the main result of this paper, combining Lemmas 6 and 8 to prove that, when ε is small, the subdomain iterates converge faster than shown in Lemma 6, and that the resulting approximation is parameter-uniform.

Theorem 1. *Let $\boldsymbol{u}$ be the solution to (1)–(2) and $\boldsymbol{U}^{[k]}(x_i)$ be the k^{th} iterate of the discrete Schwarz method described in Section 2. If $\tau = 2\varepsilon\alpha^{-1}\ln N$ and $N > 2$, then*

$$\|\boldsymbol{u} - \boldsymbol{U}^{[k]}\|_{\overline{\Omega}^N} \leq C_0 N^{-2k} + C_1 (N^{-1}\ln N)^2.$$

Proof. From Lemma 8 there exists

$$\boldsymbol{U} := \lim_{k\to\infty} \boldsymbol{U}^{[k]}.$$

We know from Lemma 6 that there exists C_2 and C_3 such that

$$\|\boldsymbol{u} - \boldsymbol{U}^{[k]}\|_{\overline{\Omega}^N} \leq C_2 2^{-k} + C_3 (N^{-1}\ln N)^2.$$

This implies that

$$\|\boldsymbol{u} - \boldsymbol{U}\|_{\overline{\Omega}^N} \leq C_1 (N^{-1}\ln N)^2.$$

We also know from Lemma 8 that there exists C_4 such that

$$\|\boldsymbol{U}^{[k+1]} - \boldsymbol{U}^{[k]}\|_{\overline{\Omega}^N} \leq C_4 N^{-2k}.$$

Consequently, for $N \geq 2$, there exists C_0 such that

$$\begin{aligned}
\|\boldsymbol{U} - \boldsymbol{U}^{[k]}\|_{\overline{\Omega}^N} &\leq C \sum_{l=k}^{\infty} N^{-2l} \\
&= C \frac{N^{-2k}}{1 - N^{-2}} \\
&\leq C_0 N^{-2k}.
\end{aligned}$$

We can thus conclude that

$$\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{U}^{[k]}\|_{\overline{\Omega}^N} &= \|\boldsymbol{U} - \boldsymbol{U}^{[k]} + \boldsymbol{u} - \boldsymbol{U}\|_{\overline{\Omega}^N} \\
&\leq \|\boldsymbol{U} - \boldsymbol{U}^{[k]}\|_{\overline{\Omega}^N} + \|\boldsymbol{u} - \boldsymbol{U}\|_{\overline{\Omega}^N} \\
&\leq C_0 N^{-2k} + C_1 (N^{-1}\ln N)^2.
\end{aligned}$$

$\square$

5 Conclusions

Note that from Theorem 1, for $k \geq 1$ the $(N^{-1}\ln N)^2$ term dominates the error bound. Thus, for small ε, the desired accuracy is attained after only one iteration.

The efficiency of this method is essentially due to the fact that the subdomain overlaps are just outside the boundary layers. While one could construct a Schwarz algorithm with multiple subdomains it is unlikely that one would observe the rapid convergence seen with this algorithm.

When the equations may have distinct singular perturbation parameters the solutions contain overlapping layers which necessitate the construction of a Schwarz method based on five overlapping layers. The authors investigate this problem in [18] and show that this new method satisfies

$$\|\boldsymbol{u} - \boldsymbol{U}^{[k]}\|_{\overline{\Omega}^N} \leq C_0 2^{-k} + C_1 (N^{-1}\ln N)^2.$$

Furthermore we show that, when the singular perturbation parameters are of different magnitudes, this bound is sharp. Consequently one does not observe the rapid convergence of the Schwarz iterates, as is the case in this paper.

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