SIMPLICIAL EMBEDDINGS BETWEEN PANTS GRAPHS

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Abstract

We prove that, except in some low-complexity cases, every locally injective simplicial map between pants graphs is induced by a π_1-injective embedding between the corresponding surfaces.

1 Introduction and main results

To a surface Σ one may associate a number of naturally defined objects – its Teichmüller space, mapping class group, curve or pants graph, etc. An obvious problem is then to study embeddings between objects in the same category, where the term "embedding" is to be interpreted suitably in each case, for instance "isometric embedding" in the case of Teichmüller spaces, "injective homomorphism" in the case of mapping class groups, and "injective simplicial map" in the case of curve and pants graphs.

For pants graphs, this problem was first studied by D. Margalit [Mar], who showed that every automorphism of the pants graph is induced by a self-homeomorphism of Σ. More concretely, let $\mathrm{Mod}(\Sigma)$ be the mapping class group of Σ, which acts on the pants graph $\mathcal{P}(\Sigma)$ by simplicial automorphisms, and let $\mathrm{Aut}(\mathcal{P}(\Sigma))$ be the group of all simplicial automorphisms of $\mathcal{P}(\Sigma)$. Let $\kappa(\Sigma)$ be the complexity of Σ, that is, the cardinality of a pants decomposition of Σ. The following is part of Theorem 1 of [Mar]:

Theorem 1 ([Mar])**.** *If Σ is a compact, connected, orientable surface with $\kappa(\Sigma) > 0$, then the natural homomorphism $\mathrm{Mod}(\Sigma) \to \mathrm{Aut}(\mathcal{P}(\Sigma))$ is surjective. Moreover, if $\kappa(\Sigma) > 3$ then it is an isomorphism.*

The main purpose of this note is to extend Margalit's result to (locally) injective simplicial maps between pants graphs. We note that examples of such maps are plentiful. Indeed, let Σ_1 be an essential subsurface of Σ_2 (see Section 2 for definitions) whose every connected component has positive complexity. Then one may construct an injective simplicial map $\phi : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ by first choosing a multicurve Q that extends any

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pants decomposition of Σ_1 to a pants decomposition of Σ_2 and then setting $\phi(v) = v \cup Q$.

Our main result asserts that, except in some low-complexity cases, this is the only way in which injective simplicial maps of pants graphs arise. Given a simplicial map $\phi : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ and a π_1-injective embedding $h : \Sigma_1 \to \Sigma_2$, we say that ϕ is *induced* by h if there exists a multicurve Q on Σ_2, disjoint from $h(\Sigma_1)$, such that $\phi(v) = h(v) \cup Q$ for all vertices v of $\mathcal{P}(\Sigma_1)$. In particular, Q has cardinality $\kappa(\Sigma_2) - \kappa(\Sigma_1)$. We will show:

Theorem A. *Let Σ_1 and Σ_2 be compact orientable surfaces of negative Euler characteristic, such that each connected component of Σ_1 has complexity at least 2. Let $\phi : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ be an injective simplicial map. Then there exists a π_1-injective embedding $h : \Sigma_1 \to \Sigma_2$ that induces ϕ.*

We note that the hypothesis that all connected components of Σ_1 have complexity at least 2 is necessary, since the pants graph of the 1-holed torus and the 4-holed sphere are isomorphic (see [Min], for instance).

Remark. In the case of curve graphs, Teichmüller spaces and mapping class groups, there exist embeddings for which there are no π_1-injective embeddings of the corresponding surfaces. First, one may construct an injective simplicial map from the curve graph of a closed surface X to that of $X - p$, by considering a point p in the complement of the union of all simple closed geodesics on X. Next, any finite-degree cover $\tilde{Y} \to Y$ gives rise to an isometric embedding $T(Y) \to T(\tilde{Y})$ of Teichmüller spaces, so we may take Y to be a closed surface in order to produce the desired example. Finally, there exist injective homomorphisms of mapping class groups with no π_1-injective embeddings between the corresponding surfaces, se [BirHi] and [ALS].

In order to prove Theorem A, we will closely follow Margalit's strategy in [Mar] for proving Theorem 1. In Section 2 we will introduce the pants graph and its natural subgraphs. In Section 3 we will study some objects in the pants graph, namely *Farey graphs* and *admissible tuples*, which appear, or at least have their origin, in [Mar]. Most importantly, the structure of these objects is preserved by injective simplicial maps. Using this, in Section 4 we will show the following result, which will constitute the main step for proving Theorem A. Given a multicurve Q on Σ, let $\mathcal{P}_Q$ be the subgraph of $\mathcal{P}(\Sigma)$ spanned by those vertices containing Q. A *non-trivial component* of Σ is a connected component of Σ of positive complexity.

Theorem B. *Let Σ_1 and Σ_2 be compact orientable surfaces such that every connected component of Σ_1 has positive complexity. Let $\phi : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ be an injective simplicial map. Then the following hold:*

(1) $\kappa(\Sigma_1) \leq \kappa(\Sigma_2)$,

(2) There exists a multicurve Q on Σ_2, of cardinality $\kappa(\Sigma_2) - \kappa(\Sigma_1)$, such that $\phi(\mathcal{P}(\Sigma_1)) = \mathcal{P}_Q$. In particular, $\mathcal{P}(\Sigma_1) \cong \mathcal{P}(\Sigma_2 - Q)$;

(3) Σ_1 and $\Sigma_2 - Q$ have the same number of non-trivial components. Moreover, if $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_r$ are, respectively, the non-trivial components of Σ_1 and $\Sigma_2 - Q$ then, up to reordering the indices, ϕ induces an isomorphism $\phi_i : \mathcal{P}(X_i) \to \mathcal{P}(Y_i)$. In particular, $\kappa(X_i) = \kappa(Y_i)$.

Theorem B itself has an interesting consequence for pants graph automorphisms. More concretely, in Corollary 10 of Section 5 we will see that pants graph automorphisms preserve the pants graph stratification (see Section 2 for definitions). This implies Theorem 1 if Σ is not the 2-holed torus. The case of the 2-holed torus needs some extra care but it also follows from Theorem B by applying the same strategy of [Mar], Section 5.

Finally, in Section 6 we will prove Theorem A, which will follow easily from Theorem B and the (folklore) classification of pants graphs up to isomorphism, included as Lemma 11 in Section 6.

We remark that, even though Theorems A and B are stated for injective simplicial maps, our arguments will only require the maps to be simplicial and *locally injective*, that is, injective on the star of every vertex of $\mathcal{P}(\Sigma_1)$. The *star* of a vertex is defined as the union of all edges incident on it. In particular, we have:

Theorem C. *Let Σ_1 and Σ_2 be compact orientable surfaces of negative Euler characteristic, such that each connected component of Σ_1 has complexity at least 2. Let $\phi : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ be a locally injective simplicial map. Then there exists a π_1-injective embedding $h : \Sigma_1 \to \Sigma_2$ that induces ϕ.*

Finally, we point out that a number of authors have studied embeddings in the context of mapping class groups and other complexes associated to surfaces. References include [ALS], [BehMa], [BelMa], [BirHi], [Irm1], [Irm2], [Irm3], [IrmKo], [IrmMc], [Iva], [IvaMc], [Ko], [Luo], [PaRo], [Sch], [Sha].

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2 Definitions and basic results

2.1 Surfaces and curves.

Let Σ be a compact orientable surface whose every connected component has negative Euler characteristic. If g and b are, respectively, the genus and number of boundary components of Σ, we will refer to the number $\kappa(\Sigma) = 3g - 3 + b$ as the *complexity* of Σ. As mentioned in Section 1, a *non-trivial component* of Σ is a connected component of Σ that has positive complexity.

A subsurface $X \subset \Sigma$ is said to be *essential* if no components of X are parallel to $\partial\Sigma$ and every component of ∂X determines either a non-homotopically trivial simple closed curve on Σ or a component of $\partial\Sigma$. Throughout this note we will only consider essential subsurfaces whose every connected component has negative Euler characteristic. Two subsurfaces are said to be *disjoint* if they can be homotoped away from each other.

A simple closed curve on Σ is said to be *peripheral* if it is homotopic to a component of $\partial\Sigma$. By a *curve* on Σ we will mean a homotopy class of non-trivial and non-peripheral simple closed curves on Σ. The *intersection number* between two curves α and β is defined as

$$i(\alpha, \beta) = \min\{|a \cap b| : a \in \alpha, b \in \beta\}.$$

If $i(\alpha, \beta) = 0$, we say that α and β are *disjoint*. A *multicurve* is a collection of distinct and disjoint curves on Σ. Given a multicurve Q on Σ, the *deficiency* of Q is defined to be $\kappa(\Sigma) - |Q|$. A *pants decomposition* is a multicurve of cardinality $\kappa(\Sigma)$ (and so maximal with respect to inclusion). Note, if Q is a pants decomposition then $\Sigma - Q$ is a disjoint union of 3-holed spheres, or *pairs of pants*.

If $X \subset \Sigma$ is a (not necessarily proper) subsurface, we say that a collection $\mathcal{A}$ of curves on X *fills* X if, for every curve γ on X, there exists $\alpha \in \mathcal{A}$ with $i(\alpha, \gamma) > 0$. In particular, if $\kappa(X) = 1$ then any pair of distinct curves fill X.

2.2 The pants graph.

We say that two pants decompositions are related by an *elementary move* if they have a deficiency 1 multicurve in common, and the remaining two curves either fill a 4-holed sphere and intersect exactly twice, or they fill a 1-holed torus and intersect exactly once. See Figure 1.

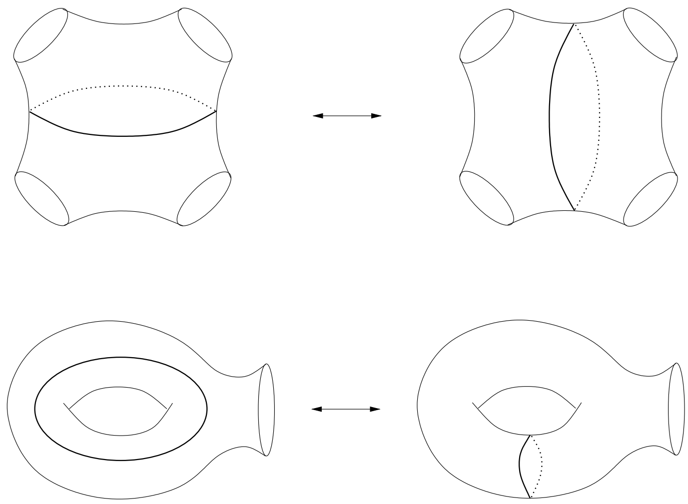

Figure 1: The two types of elementary move.

The *pants graph* $\mathcal{P}(\Sigma)$ of Σ is the simplicial graph whose vertex set is the set of all pants decompositions of Σ and where two vertices are connected by an edge if the corresponding pants decompositions are related by an elementary move. A *path* in $\mathcal{P}(\Sigma)$ is a sequence $v_1, \ldots, v_n$ of adjacent vertices of $\mathcal{P}(\Sigma)$. A *circuit* is a path $v_1, \ldots, v_n$ such that $v_1 = v_n$ and $v_i \neq v_j$ for all other i, j.

The pants graph was introduced by Hatcher-Thurston in [HatTh], who proved it is connected (see the remark on the last page of [HatTh]). A detailed proof was then given by Hatcher-Lochak-Schneps in [HLS], where they proved that attaching 2-cells to finitely many types of circuits in $\mathcal{P}(\Sigma)$ produces a simply-connected 2-complex, known as the *pants complex*. The graph $\mathcal{P}(\Sigma)$ becomes a geodesic metric space by declaring each edge to have length 1, and Brock [Br] recently showed that $\mathcal{P}(\Sigma)$ is quasi-isometric to the Weil-Petersson metric on the Teichmüller space of Σ.

2.3 Natural subgraphs.

As mentioned in the introduction, if $Y \subset \Sigma$ is an essential subsurface with no trivial components, then the inclusion map induces an injective simplicial map $\mathcal{P}(Y) \to \mathcal{P}(\Sigma)$, and so we can regard $\mathcal{P}(Y)$ as a connected subgraph of $\mathcal{P}(\Sigma)$. If $Y_1, Y_2 \subset \Sigma$ are disjoint essential subsurfaces of positive complexity, then $\mathcal{P}(Y_1) \times \mathcal{P}(Y_2)$, the 1-skeleton of the product of $\mathcal{P}(Y_1)$ and $\mathcal{P}(Y_2)$, is a connected subgraph of $\mathcal{P}(\Sigma)$. Moreover, if Σ is not connected, then $\mathcal{P}(\Sigma)$ is the 1-skeleton of the product of the pants graphs of its non-trivial components.

Given a multicurve Q on Σ, let $\mathcal{P}_Q$ be the subgraph of $\mathcal{P}(\Sigma)$ spanned by those vertices of $\mathcal{P}(\Sigma)$ that contain Q. It will be convenient to consider the empty set as a multicurve, in which case we set $\mathcal{P}_\emptyset$ to be equal to $\mathcal{P}(\Sigma)$. Note that $\mathcal{P}_Q$ is connected for all multicurves Q; indeed, if Q is strictly contained in a pants decomposition then $\mathcal{P}_Q$ is naturally isomorphic to $\mathcal{P}(\Sigma - Q)$, and if Q is itself a pants decomposition then $\mathcal{P}_Q$ is equal to Q.

If Q_1 and Q_2 are multicurves on Σ, then $\mathcal{P}_{Q_1} \cap \mathcal{P}_{Q_2} \neq \emptyset$ if and only if $Q_1 \cup Q_2$ is a multicurve, in which case $\mathcal{P}_{Q_1} \cap \mathcal{P}_{Q_2} = \mathcal{P}_{Q_1 \cup Q_2}$. Furthermore, $\mathcal{P}_{Q_1} \subset \mathcal{P}_{Q_2}$ if and only if $Q_2 \subset Q_1$. This endows the pants graph with a *stratified structure*, analogous to the stratification of the Weil-Petersson completion (see [Wol]), with strata all the subgraphs of the form $\mathcal{P}_Q$, for some multicurve Q. Then $\mathcal{P}(\Sigma)$ is the union of all strata, and two strata intersect over a stratum if at all.

3 Some objects in the pants graph.

3.1 Farey graphs.

The *standard Farey graph* is the simplicial graph whose vertex set is $\mathbb{Q} \cup \{\infty\}$ and where two vertices p/q and r/s, in lowest terms, are connected by an edge if $|ps - rq| = 1$. It is usually represented as an ideal triangulation of the Poincaré disc model of the hyperbolic plane. By a *Farey graph* we will mean an isomorphic copy of the standard Farey graph. The following result is implicit in the proof of Lemma 1 of [Mar].

Lemma 2 (Structure of Farey graphs)**.** *A subgraph F of $\mathcal{P}(\Sigma)$ is a Farey graph if and only if $F = \mathcal{P}_Q$, for some deficiency 1 multicurve Q.*

Proof. Consider $\mathcal{P}_Q$, where Q is a deficiency 1 multicurve. Then $\Sigma - Q$ has a unique non-trivial component X, which has complexity 1, and so it is either a 1-holed torus or a 4-holed sphere. In either case, $\mathcal{P}(X)$ is a Farey graph (see, for instance, [Min], Section 3), and $\mathcal{P}_Q \cong \mathcal{P}(\Sigma - Q) \cong \mathcal{P}(X)$.

Conversely, if $\Delta \subset \mathcal{P}(\Sigma)$ is a circuit of length 3, then $\Delta \subset \mathcal{P}_Q$ for some deficiency 1 multicurve Q, by the definition of adjacency in $\mathcal{P}(\Sigma)$. The result now follows easily from the observation that any two vertices of a Farey graph can be connected by a sequence of circuits of length 3 such that any two consecutive such circuits have exactly two vertices in common. $\square$

In the situation of Lemma 2, we will say that F is *determined* by Q. Let e be an edge of $\mathcal{P}(\Sigma)$, and let u and v be its endpoints. Then e is contained in a unique Farey graph, determined by the deficiency 1 multicurve $u \cap v$. Given a vertex u of $\mathcal{P}(\Sigma)$, observe that there are exactly $\kappa(\Sigma)$ distinct Farey graphs containing u, determined by the $\kappa(\Sigma)$ distinct deficiency 1 multicurves contained in u. We state this observation as a separate lemma, as we will make extensive use of it later.

Lemma 3. *Given any vertex u of $\mathcal{P}(\Sigma)$, there are exactly $\kappa(\Sigma)$ distinct Farey graphs containing u.* $\square$

As mentioned in Section 1, the *star* $\mathrm{St}(u)$ of a vertex u of $\mathcal{P}(\Sigma)$ is the union of all edges of $\mathcal{P}(\Sigma)$ incident on u. By the discussion preceding Lemma 3, each edge of $\mathrm{St}(u)$ is contained in exactly one of $\kappa(\Sigma)$ Farey graphs. The following remark offers a characterisation of when two edges of $\mathrm{St}(u)$ are contained in the same Farey graph, and makes apparent that such property is preserved by locally injective simplicial maps. The proof is immediate.

Lemma 4. *Let u be a vertex of $\mathcal{P}(\Sigma)$. Two edges $e, e' \in \mathrm{St}(u)$ are contained in the same Farey graph if and only if there exists a sequence of edges $e = e_0, e_1, \dots, e_n = e'$ in $\mathrm{St}(u)$ such that e_i and e_{i+1} are edges of the same circuit of length 3 in $\mathcal{P}(\Sigma)$, for all $i = 0, \dots, n-1$.* $\square$

We end this subsection with the following observation, which asserts that if a Farey graph F intersects a stratum in the pants graph, then either F is contained in the stratum or else is "transversal" to it.

Lemma 5. *Let F be a Farey graph in $\mathcal{P}(\Sigma)$ and let T be a multicurve. Suppose that $F \cap \mathcal{P}_T$ has at least 2 vertices. Then $F \subseteq \mathcal{P}_T$.*

Proof. Lemma 2 implies that $F = \mathcal{P}_Q$, for some deficiency 1 multicurve Q. Suppose there exist two distinct vertices u, v in $\mathcal{P}_Q \cap \mathcal{P}_T$. Write $u = Q \cup \alpha$, $v = Q \cup \beta$, noting $\alpha \neq \beta$. Since $u, v \in \mathcal{P}_T$ then $T \subseteq Q$, and therefore $\mathcal{P}_Q \subseteq \mathcal{P}_T$. $\square$

3.2 Admissible tuples in the pants graph.

We now introduce the notion of *admissible tuple* in the pants graph, a slight generalisation of what Margalit refers to as "alternating circuit" in [Mar].

Definition 6 (Admissible tuple). *Let $n > 3$ and let $(v_1, \ldots, v_n)$ be a cyclically ordered n-tuple of distinct vertices of $\mathcal{P}(\Sigma)$. We say that $(v_1, \ldots, v_n)$ is* admissible *if v_i and v_{i+1} belong to the same Farey graph F_i, and $F_i \neq F_{i+1}$ (counting subindices modulo n).*

In particular $F_i \cap F_{i+1} = \{v_{i+1}\}$. An admissible 5-tuple will be called a *pentagon* if v_i and v_{i+1} are adjacent for all i. The following lemma describes the structure of admissible 4- and 5-tuples in the pants graph, and will be crucial in the proof of our main results. We remark that this result is implicit in [Mar].

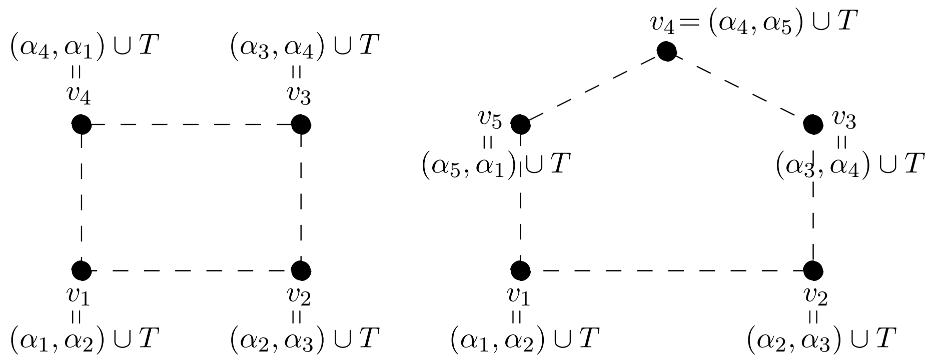

Figure 2: Admissible 4- and 5-tuples in the pants graph. Here T is a deficiency 2 multicurve, and the dashed line between v_i and v_{i+1} represents a path between them, entirely contained in a Farey graph F_i.

Lemma 7 (Structure of admissible 4- and 5-tuples). *Let $(v_1, \ldots, v_n)$ be an admissible n-tuple, where $n \in \{4, 5\}$. Then there exists a deficiency 2 multicurve T such that $v_i \in \mathcal{P}_T$ for all i. Moreover, if $n = 4$ then $\Sigma - T$ has exactly 2 non-trivial components, each of complexity 1; if $n = 5$ then $\Sigma - T$ has exactly 1 non-trivial component, which has complexity 2.*

Proof. For the first part, note there is nothing to show if $\kappa(\Sigma) = 2$, for in that case we let $T = \emptyset$, so that $\mathcal{P}_T = \mathcal{P}(\Sigma)$. So assume $\kappa(\Sigma) \geq 3$. Since v_1, v_2, v_3 do not belong to the same Farey graph, then $T = v_1 \cap v_2 \cap v_3$ is a deficiency 2 multicurve. Now $T \subset v_4$ as well; otherwise v_1 and v_4 would differ by 3 curves and thus one could not connect v_1 and v_4 by a path entirely contained in at most 2 Farey graphs. Similarly, $T \subset v_5$ in the case of an admissible 5-tuple, and so the first part of the result follows.

Note that, in particular, one can write $v_i = (\alpha_i, \alpha_{i+1}) \cup T$ for all i, as in Figure 2. Since T has deficiency 2, then $\Sigma - T$ either has one non-trivial component of complexity 2, or two non-trivial components of complexity 1. Let X (resp. Y) be the complexity 1 subsurface filled by α_1 and α_3 (resp. α_2 and α_4), noting $X \neq Y$ and $X, Y \subset \Sigma - T$.

If $n = 4$ then $i(\alpha_j, \alpha_2) = i(\alpha_j, \alpha_4) = 0$ for $j \in \{1, 3\}$ (see Figure 2). Thus X and Y are disjoint and thus the result follows.

Now assume $n = 5$. We claim that $X \cup Y$ is connected. If not, then α_5 is contained in either X or Y. But α_5 is distinct and disjoint from both α_1 and α_4 (see Figure 2), and so α_5 cannot be contained in either X or Y, which is a contradiction. $\square$

Let F_1, F_2 be distinct Farey graphs of $\mathcal{P}(\Sigma)$ intersecting at a vertex u, where F_i is determined by the deficiency 1 multicurve Q_i, for $i = 1, 2$. If the non-trivial components of $\Sigma - Q_1$ and $\Sigma - Q_2$ are disjoint, we say that F_1 and F_2 *commute*. Now, Lemma 7 implies that if $(v_1, \ldots, v_4)$ is an admissible 4-tuple, then F_i and F_{i+1} commute for all i, where F_i is the Farey graph containing v_i and v_{i+1}. The following converse is an immediate consequence of the proof of Lemma 7:

Corollary 8. *Let F_1, F_2 be Farey graphs that commute. For any $u_i \in F_i - \{u\}$, the vertices u, u_1, u_2 are elements of an admissible 4-tuple.*

Proof. Let $u_i \in F_i - \{u\}$ for $i = 1, 2$. Since $F_1 \neq F_2$ then $u = (\alpha_1, \alpha_2) \cup T$, $u_1 = (\alpha'_1, \alpha_2) \cup T$ and $u_2 = (\alpha_1, \alpha'_2) \cup T$, for some deficiency 2 multicurve T. Now F_1 and F_2 commute, and so $i(\alpha'_1, \alpha'_2) = 0$. Therefore, (u_2, u, u_1, w) is an admissible 4-tuple, where $w = (\alpha'_1, \alpha'_2) \cup T$. $\square$

In particular, there exists an admissible 4-tuple in $\mathcal{P}(\Sigma)$ if and only if $\kappa(\Sigma) \geq 3$. The next technical result will be very important in the next section:

Lemma 9 (Extending adjacent vertices to admissible tuples)**.** *Let Σ be a surface of complexity at least 2. Let u and v be adjacent vertices of $\mathcal{P}(\Sigma)$ and let G be a Farey graph containing v but not u. Then there exists $n \in \{4, 5\}$ and a vertex $w \in G - \{v\}$, such that u, v, w are elements of an admissible n-tuple.*

Proof. Write $k = \kappa(\Sigma)$, $u = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $v = (\alpha'_1, \alpha_2, \ldots, \alpha_k)$. Let F be the Farey graph containing u and v, and hence determined by

$u \cap v = v - \alpha_1'$. Since $u \notin G$ then, up to relabeling the curves of v, G is determined by $v - \alpha_2$. Let T denote the deficiency 2 multicurve $v - (\alpha_1', \alpha_2)$. There are two cases to consider:

Case 1. *F and G commute.* In this case the result follows from Corollary 8 by considering any $w \in G$, with $w \neq v$.

Case 2. *F and G do not commute.* Then $\Sigma - T$ has exactly one non-trivial component S, of complexity 2, and (α_1, α_2) and (α_1', α_2) are adjacent vertices of $\mathcal{P}(S)$. There are two possibilities for S, namely S is a 5-holed sphere or S is a 2-holed torus.

If S is a 5-holed sphere then, up to the action of $\mathrm{Mod}(S)$, the curves $\alpha_1, \alpha_2, \alpha_1'$ are, respectively, the curves α, β, γ on the left of Figure 3. Consider the curves δ and η, also from the left of Figure 3, and set $v_i = w_i \cup T$, where w_i is defined as in Figure 3. Then $(v_1, \ldots, v_5)$ is an admissible 5-tuple (in this case, a pentagon) in $\mathcal{P}(\Sigma)$, noting $v_1 = u$, $v_2 = v$ and $v_3 \in G$. Thus we can take $w = v_3$ and so the result follows.

The case of S a 2-holed torus is dealt with along the exact same lines, using the curves on the 2-holed torus of Figure 3; in this case, the 5-tuple we obtain is not longer a pentagon (in fact, there are no pentagons in the pants graph of the 2-holed torus; see the proof of Lemma 8 in [Mar]) □

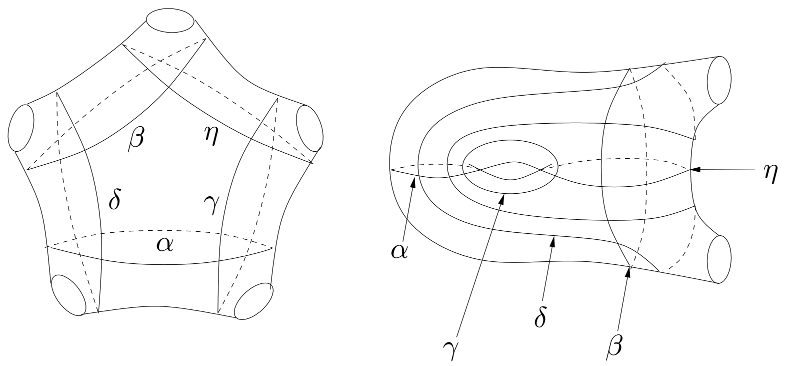

Figure 3: Curves giving rise to an admissible 5-tuple $(w_1, \ldots, w_5)$ in a 5-holed sphere (left) and a 2-holed torus (right), where $w_1 = (\alpha, \beta)$, $w_2 = (\beta, \gamma)$, $w_3 = (\gamma, \delta)$, $w_4 = (\delta, \eta)$ and $w_5 = (\eta, \alpha)$.

4 Proof of Theorem B

The main ingredient in the proof of Theorem B will be that (locally) injective simplicial maps of pants graphs preserve Farey graphs and admissible tuples. Let us briefly comment on this. First, a quick argument involving the dual tree of a Farey graph shows that a (locally) injective simplicial map of a Farey graph to itself is in fact bijective; this argument is recorded in Lemma 15 of [Sha], and we do not include it here.

Let $\phi : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ be a (locally) injective simplicial map. By the discussion above, if F is a Farey graph, then so is $\phi(F)$. Now, Lemma 4 implies that if F, F' are distinct Farey graphs that intersect at a vertex, then the same is true for the Farey graphs $\phi(F)$ and $\phi(F')$. Using this, plus the definition of admissible tuple, we get that ϕ maps admissible n-tuples to admissible n-tuples.

We can now prove Theorem B. Recall the statement:

Theorem B. *Let Σ_1 and Σ_2 be compact orientable surfaces such that every connected component of Σ_1 has positive complexity. Let $\phi : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ be an injective simplicial map. Then the following hold:*

(1) $\kappa(\Sigma_1) \leq \kappa(\Sigma_2)$,

(2) There exists a multicurve Q on Σ_2, of cardinality $\kappa(\Sigma_2) - \kappa(\Sigma_1)$, such that $\phi(\mathcal{P}(\Sigma_1)) = \mathcal{P}_Q$. In particular, $\mathcal{P}(\Sigma_1) \cong \mathcal{P}(\Sigma_2 - Q)$;

(3) Σ_1 and $\Sigma_2 - Q$ have the same number of non-trivial components. Moreover, if $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_r$ are, respectively, the non-trivial components of Σ_1 and $\Sigma_2 - Q$ then, up to reordering the indices, ϕ induces an isomorphism $\phi_i : \mathcal{P}(X_i) \to \mathcal{P}(Y_i)$. In particular, $\kappa(X_i) = \kappa(Y_i)$.

Proof. Observe that if $\kappa(\Sigma_1) = 1$ then the result follows from Lemma 2. Therefore, from now on we will assume that $\kappa(\Sigma_1) \geq 2$. Let $\kappa_i = \kappa(\Sigma_i)$, for $i = 1, 2$.

Part (1) is immediate, since we know ϕ maps distinct Farey graphs containing a vertex u (there are κ_1 of these, by Lemma 3) to distinct Farey graphs containing $\phi(u)$ (there are κ_2 of these). We will now prove part (2). For clarity, its proof will be broken down into 3 separate claims.

Claim I. *Let u be a vertex of $\mathcal{P}(\Sigma_1)$. There exists a multicurve $Q(u)$ on Σ_2, of cardinality $\kappa_2 - \kappa_1$, such that $\phi(e) \subset \mathcal{P}_{Q(u)}$ for all $e \in \mathrm{St}(u)$.*

Proof. By Lemma 3, there are κ_1 Farey graphs $F_1, \ldots, F_{\kappa_1}$ containing u. By Lemma 2, $\phi(F_i) = \mathcal{P}_{Q_i}$, for some deficiency 1 multicurve $Q_i \subset \phi(u)$. Consider the multicurve $Q(u) = Q_1 \cap \cdots \cap Q_{\kappa_1}$, which has cardinality $\kappa_2 - \kappa_1$. Since $Q(u) \subset Q_i$ then $\phi(F_i) = \mathcal{P}_{Q_i} \subset \mathcal{P}_{Q(u)}$. In particular, $\phi(e) \subset \mathcal{P}_{Q(u)}$ for all $e \in \mathrm{St}(u)$, since $e \subset F_i$ for some i. $\diamond$

Claim II. *If v is adjacent to u, then $\phi(e') \subset \mathcal{P}_{Q(u)}$ for all $e' \in \mathrm{St}(v)$, where $Q(u)$ is the multicurve given by Claim I for u.*

Proof. Let e be the edge with endpoints u and v. Let $e' \in \mathrm{St}(v)$ and let F be the unique Farey graph containing e'. If $e \subset F$ then the result follows from the proof of Claim I. So suppose e is not contained in F, so $u \notin F$. By Lemma 9, there exist a number $n \in \{4, 5\}$ and a vertex $w \in F$, with $w \neq v$, such that u, v, w are elements of an admissible n-tuple in $\mathcal{P}(\Sigma_1)$, which we denote by τ. Thus $\phi(u), \phi(v), \phi(w)$ are also elements of an admissible n-tuple in $\mathcal{P}(\Sigma_2)$. Therefore there is a deficiency 2 multicurve T on Σ_2 such that $\phi(\tau) \subset \mathcal{P}_T$, using Lemma 7.

We now claim that $Q(u) \subseteq T$. To see this, let z be the unique element of $\tau - \{v\}$ which is contained in the same Farey graph as u, noting z, u, v are not contained in the same Farey graph of $\mathcal{P}(\Sigma_1)$ by the definition of admissible tuple. Therefore $\phi(z), \phi(u), \phi(v)$ are not contained in the same Farey graph of $\mathcal{P}(\Sigma_2)$ and so $\phi(z) \cap \phi(u) \cap \phi(v) = T$, since $\phi(\tau) \subset \mathcal{P}_T$ and T has deficiency 2. Finally, $Q(u) \subseteq \phi(z) \cap \phi(u) \cap \phi(v)$ since ϕ maps every Farey graph containing u into $\mathcal{P}_{Q(u)}$ and u, v (resp. u, z) are contained in a common Farey graph. Thus $Q(u) \subseteq T$, as desired.

Since $Q(u) \subseteq T$ then $\phi(\tau) \subset \mathcal{P}_T \subseteq \mathcal{P}_{Q(u)}$. In particular, $\phi(w)$ is contained in $\mathcal{P}_{Q(u)}$ and thus in $\phi(F) \cap \mathcal{P}_{Q(u)}$. Since $\phi(v) \in \phi(F) \cap \mathcal{P}_{Q(u)}$ as well, we conclude that $\phi(F) \subset \mathcal{P}_{Q(u)}$ by Lemma 10. In particular, $\phi(e') \subset \mathcal{P}_{Q(u)}$ and thus Claim II follows. $\diamond$

As a consequence, and since $\mathcal{P}(\Sigma_1)$ is connected, it follows that $\phi(\mathcal{P}(\Sigma_1)) \subseteq \mathcal{P}_Q$, where $Q = Q(u)$ for some, and hence any, vertex u of $\mathcal{P}(\Sigma_1)$. Actually, more is true:

Claim III. $\phi(\mathcal{P}(\Sigma_1)) = \mathcal{P}_Q$.

Proof. Let e be an edge of $\mathcal{P}_Q$; we want to show that $e \in \mathrm{Im}(\phi)$. Since $\phi(\mathcal{P}(\Sigma_1))$ and $\mathcal{P}_Q$ are connected, and since $\phi(\mathcal{P}(\Sigma_1)) \subseteq \mathcal{P}_Q$, we can assume $e \in \mathrm{St}(\phi(u))$ for some vertex u of $\mathcal{P}(\Sigma_1)$. Note that e is contained in a unique Farey graph H and that $H \subset \mathcal{P}_Q$ by Lemma 10.

Since $\mathcal{P}_Q \cong \mathcal{P}(\Sigma_2 - Q)$ and $\kappa_1 = \kappa(\Sigma_2 - Q)$, there are exactly κ_1 distinct Farey graphs in $\mathcal{P}(\Sigma_2)$ which are contained in $\mathcal{P}_Q$ and contain $\phi(u)$, by Lemma 3. Again by Lemma 3, there are exactly κ_1 distinct Farey graphs in $\mathcal{P}(\Sigma_1)$ containing u. Since ϕ maps distinct Farey graphs containing u to distinct Farey graphs containing $\phi(u)$, we get that $H = \phi(F)$ for some Farey graph F in $\mathcal{P}(\Sigma_1)$ containing u. In particular, $e \in \mathrm{Im}(\phi)$, as desired. $\diamond$

Finally, we will prove part (3). Let $X_1, \ldots, X_r$ be the non-trivial components of Σ_1. Observe that every pants decomposition of Σ_1 has the form $(v_1, \ldots, v_r)$, where v_i is a pants decomposition of X_i, and so $\mathcal{P}(\Sigma_1) = \Pi_{i=1}^r \mathcal{P}(X_i)$. Fix a pants decomposition $v = (v_1, \ldots, v_r)$ of Σ_1. Then ϕ induces an injective simplicial map

$$\phi_i : \mathcal{P}(X_i) \to \mathcal{P}_Q \cong \mathcal{P}(\Sigma_2 - Q),$$

by setting $\phi_i(w) = \phi(v_1, \ldots, v_{i-1}, w, v_{i+1}, \ldots, v_r)$ for all vertices w of X_i. Applying part (2) of Theorem B to ϕ_i, we deduce that there exists an essential subsurface Y_i of $\Sigma_2 - Q$ such that $\phi_i(\mathcal{P}(X_i)) = \mathcal{P}(Y_i)$. In particular, $\kappa(Y_i) = \kappa(X_i)$, by part (1). Moreover, by discarding those connected components of Y_i homeomorphic to a 3-holed sphere, we can assume that Y_i has no trivial components.

Claim. Y_i is connected.

Proof. Suppose, for contradiction, that Y_i had $N \geq 2$ components $Z_1, \ldots, Z_N$. In particular $0 < \kappa(Z_j) < \kappa(Y_i) = \kappa(X_i)$, for all j, and $\phi_i(\mathcal{P}(X_i)) = \mathcal{P}(Z_1) \times \ldots \times \mathcal{P}(Z_N)$. Thus, the image of an edge of $\mathcal{P}(X_i)$ under ϕ_i is contained in one of the factors above, and thus the same holds for the image of any Farey graph under ϕ_i, by Corollary 8. Moreover, if F and F' do not commute, then $\phi_i(F)$ and $\phi_i(F')$ are contained in the same factor, also by Corollary 8.

Let u be a vertex of $\mathcal{P}(X_i)$. We now define the *adjacency graph* Γ of u, introduced independently by Behrstock-Margalit [BehMa] and Shackleton [Sha]. The vertices of Γ are exactly those curves in u, and two distinct curves are adjacent in Γ if they are boundary components of the same pair of pants determined by u. Observe that Γ is connected since X_i is.

Now a Farey graph containing u is is determined by a deficiency 1 multicurve contained in u or, equivalently, by a curve in u. Moreover, two curves in u are adjacent if and only if the Farey graphs they determine do not commute. Let $\mathcal{F}$ be the graph whose vertices are those Farey graphs containing u and whose edges correspond to distinct non-commuting Farey graphs. Note $\mathcal{F}$ is isomorphic to Γ and so it is connected.

By Lemma 3 and since $\mathcal{F}$ is connected, there exist $\kappa(X_i)$ Farey graphs in $\mathcal{P}(X_i)$, all containing u, which are mapped into the same factor of $\mathcal{P}(Z_1) \times$

$\ldots \times \mathcal{P}(Z_N)$ under ϕ_i. This contradicts Lemma 3, since $\kappa(Z_j) < \kappa(X_i)$ for all j, and thus the claim follows. $\diamond$

The discussion above implies that there are r connected subsurfaces $Y_1, \ldots, Y_r$ of $\Sigma_2 - Q$ such that, up to reordering, ϕ induces an isomorphism $\phi_i : \mathcal{P}(X_i) \to \mathcal{P}(Y_i)$ for $i = 1, \ldots, r$. In particular, $\kappa(X_i) = \kappa(Y_i)$. Now,

$$\Sigma_1 = X_1 \sqcup \ldots \sqcup X_r$$

and

$$\Sigma_2 - Q \supseteq Y_1 \cup \ldots \cup Y_r,$$

and therefore the Y_i's are pairwise disjoint, since the X_i's are pairwise disjoint, $\kappa(X_i) = \kappa(Y_i)$ and $\kappa(\Sigma_1) = \kappa(\Sigma_2 - Q)$. For the same reason, they are the only non-trivial connected components of $\Sigma_2 - Q$. This finishes the proof of Part (3) of Theorem B. $\square$

5 A consequence of Theorem B

We now present an application of Theorem B to pants graph automorphisms. Let $\phi : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ be an injective simplicial map; by Theorem B, ϕ is in fact an isomorphism. Let α be a curve on Σ, and observe that $\mathcal{P}(\Sigma - \alpha) \cong \mathcal{P}_\alpha \subset \mathcal{P}(\Sigma)$. Then ϕ induces an injective simplicial map, which we also denote by ϕ, from $\mathcal{P}_\alpha$ to $\mathcal{P}(\Sigma)$. Applying Theorem B to $\Sigma_1 = \Sigma - \alpha$ and $\Sigma_2 = \Sigma$, we readily obtain the following corollary, which implies that pants graph automorphisms preserve the pants graph stratification:

Corollary 10. *Let $\phi : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ be an automorphism. Then, for every curve α, there exists a unique curve β such that $\phi(\mathcal{P}_\alpha) = \mathcal{P}_\beta$. Moreover, $\Sigma - \alpha$ and $\Sigma - \beta$ have the same number of non-trivial components.*

In [Mar], Margalit introduced the notion of a *marked* Farey graph in the pants graph. As Farey graphs, they are preserved by pants graph automorphisms. A marked Farey graph singles out exactly one curve on Σ, although there are infinitely many marked Farey graphs in $\mathcal{P}(\Sigma)$ that single out a given curve. Margalit associates, to the pants graph automorphism ϕ, a curve graph automorphism ψ by defining $\psi(\alpha)$ to be the curve β singled out by $\phi(F)$, where F is a marked Farey graph that singles out α. One of the main steps in [Mar] is to show that this construction gives rise to a well-defined map between the pants graph automorphism group and the curve graph automorphism group, which Margalit then shows is an isomorphism. If Σ is not a 2-holed torus, Theorem 1 then follows from results of Ivanov [Iva], Korkmaz [Ko] and Luo [Luo] on the automorphism group of the curve graph. The case of the 2-holed torus requires separate treatment in [Mar], and boils down to showing that the curve graph automorphism induced by a pants graph automorphism maps (non-)separating curves to (non-)separating curves.

Similarly, one could define a curve graph automorphism ψ from the pants graph automorphism ϕ, by setting $\psi(\alpha) = \beta$, where β is the curve such that $\phi(\mathcal{P}_\alpha) = \mathcal{P}_\beta$ in Corollary 10. One quickly checks that this produces an isomorphism between the pants graph automorphism group and the curve graph automorphism group, and thus Theorem 1 follows if the surface is not the 2-holed torus. The case of the 2-holed torus is also deduced from Corollary 10 by applying the exact same argument as in [Mar]. We remark that this approach to pants graph automorphisms is similar in spirit to those of Masur-Wolf [MasWo] and Brock-Margalit [BrMa] for showing that Weil-Petersson isometries are induced by surface self-homeomorphisms. Indeed, one of the key steps there is to prove that Weil-Petersson isometries preserve the stratification of the Weil-Petersson completion.

6 Proof of Theorem A

We are finally ready to give a proof of Theorem A. We will need the following lemma, which we believe is folklore, but which we nevertheless prove for completeness.

Lemma 11 (Classification of pants graphs up to isomorphism)**.** *Let Σ, Σ' be two compact connected orientable surfaces of complexity at least 2. Then $\mathcal{P}(\Sigma)$ and $\mathcal{P}(\Sigma')$ are isomorphic if and only if Σ and Σ' are homeomorphic.*

Proof. First, by Part (1) of Theorem B, if $\mathcal{P}(\Sigma)$ and $\mathcal{P}(\Sigma')$ are isomorphic then $\kappa(\Sigma) = \kappa(\Sigma')$. We consider the following three cases:

(i) Suppose $\kappa(\Sigma) > 3$. By Theorem 1, $\mathrm{Aut}(\mathcal{P}(\Sigma)) \cong \mathrm{Mod}(\Sigma)$. Thus if $\mathcal{P}(\Sigma) \cong \mathcal{P}(\Sigma')$ then $\mathrm{Mod}(\Sigma) \cong \mathrm{Mod}(\Sigma')$ and thus Σ and Σ' are homeomorphic by Theorem 2 of [Sha]. (We remark that Shackleton's result was first proved by Ivanov-McCarthy [IvaMc] for surfaces of positive genus.)

(ii) Suppose that $\kappa(\Sigma) = 2$. Up to renaming the surfaces, Σ is a 5-holed sphere and Σ' is a 2-holed torus. The curves on the 5-holed sphere on the left of Figure 3 yield the existence of a pentagon in $\mathcal{P}(\Sigma)$, while there are no pentagons in the pants graph of the 2-holed torus (see the proof of Lemma 8 in [Mar]).

(iii) Finally, we consider the case $\kappa(\Sigma) = 3$. Let us denote by $S_{g,b}$ the surface of genus g with b boundary components. We have that

$$(\Sigma, \Sigma') \in \{(S_{0,6}, S_{2,0}), (S_{1,3}, S_{0,6}), (S_{1,3}, S_{2,0})\}$$

Suppose, for contradiction, that there exists an isomorphism ϕ between $\mathcal{P}(S_{0,6})$ and $\mathcal{P}(S_{2,0})$. Choose a curve α on $S_{0,6}$ such that $S_{0,6} - \alpha = S_{0,3} \sqcup S_{0,5}$, noting that $\mathcal{P}_\alpha \cong \mathcal{P}(S_{0,5})$. By Theorem B, there exists a curve β on $S_{2,0}$ such that $\phi(\mathcal{P}_\alpha) = \mathcal{P}_\beta$. Moreover, $S_{2,0} - \beta$ has to be connected, and thus $S_{2,0} - \beta$ is homeomorphic to $S_{1,2}$. In particular, $\mathcal{P}(S_{2,0} - \beta) \cong \mathcal{P}(S_{1,2})$. Thus we get an isomorphism between $\mathcal{P}(S_{0,5})$ and $\mathcal{P}(S_{1,2})$, which contradicts (ii).

Now, suppose there is an isomorphism ϕ between $\mathcal{P}(S_{1,3})$ and $\mathcal{P}(S_{0,6})$. We choose a curve α on $S_{1,3}$ such that $S_{1,3} - \alpha = S_{0,3} \sqcup S_{1,2}$. Arguing as above, we get an isomorphism between $\mathcal{P}(S_{1,2})$ and $\mathcal{P}(S_{0,5})$, which does not exist, by (ii).

Finally, suppose that there exists an isomorphism between $\mathcal{P}(S_{1,3})$ and $\mathcal{P}(S_{2,0})$. Choose a non-separating curve γ on $S_{1,3}$, so that $S_{1,3} - \gamma = S_{0,5}$. By the same argument as above, we get an isomorphism between $\mathcal{P}(S_{0,5})$ and $\mathcal{P}(S_{1,2})$, contradicting (ii). $\square$.

As mentioned before, the pants graph of the two complexity 1 surfaces (that is, the 1-holed torus and the 4-holed sphere) are isomorphic. We remark that the isomorphism classification of pants graphs is slightly different to that of curve complexes (see Lemma 2.1 in [Luo]).

Proof of Theorem A. Let $\phi : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ be an injective simplicial map. By Theorem B, there exists a multicurve Q on Σ_2, of deficiency $\kappa(\Sigma_1)$, such that $\phi(\mathcal{P}(\Sigma_1)) = \mathcal{P}_Q \cong \mathcal{P}(\Sigma_2 - Q)$. Discarding the trivial components of $\Sigma_2 - Q$ we obtain an essential subsurface $Y \subset \Sigma_2 - Q$, with no trivial components, and such that $\mathcal{P}(\Sigma_1) \cong \mathcal{P}(Y)$. We can thus view ϕ as an isomorphism $\mathcal{P}(\Sigma_1) \to \mathcal{P}(Y)$. Let us first assume that Σ_1 is connected. In that case Y is connected as well, by part (3) of Theorem B. Since $\kappa(\Sigma_1) \geq 2$, Lemma 11 implies there exists a homeomorphism $g : \Sigma_1 \to Y$, which induces an isomorphism $\psi : \mathcal{P}(\Sigma_1) \to \mathcal{P}(Y)$ by $\psi(v) = g(v)$. By Theorem 1, there exists $f \in \mathrm{Mod}(Y)$ such that $\phi = f \circ \psi$. Thus $f \circ g$ induces ϕ.

If Σ_1 is not connected, the result follows by applying the above argument to each connected component of Σ_1. $\square$

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