

# COXETER ARRANGEMENTS AND SOLOMON'S DESCENT ALGEBRA

J. MATTHEW DOUGLASS, GÖTZ PFEIFFER, AND GERHARD RÖHRLE

ABSTRACT. We refine a conjecture by Lehrer and Solomon on the structure of the Orlik–Solomon algebra of a finite Coxeter group  $W$  and relate it to the descent algebra of  $W$ . As a result, we claim that both the group algebra of  $W$ , as well as the Orlik–Solomon algebra of  $W$  can be decomposed into a sum of induced one-dimensional representations of element centralizers, one for each conjugacy class of elements of  $W$ . We give a uniform proof of the claim for symmetric groups.

## 1. INTRODUCTION

Suppose  $V$  is a finite-dimensional, complex vector space. A linear transformation  $t$  in  $\mathrm{GL}(V)$  is called a *reflection* if the fixed point set of  $t$  is a hyperplane in  $V$ , or equivalently, the 1-eigenspace of  $t$  has codimension 1. Suppose that  $W \subset \mathrm{GL}(V)$  is a finite Coxeter group with Coxeter generating set  $S$ . Then each  $s$  in  $S$  acts on  $V$  as a reflection with order two and  $W$  is generated by  $S$  subject to the relations  $(st)^{m_{s,t}} = 1$ , where  $m_{s,s} = 1$  and  $m_{s,t} = m_{t,s} > 1$  for  $s \neq t$  in  $S$ . Let  $T$  denote the set of all reflections in  $W$ . For each  $t$  in  $T$ , let  $H_t = \mathrm{Fix}(t)$  denote the fixed point set of  $t$  and set  $\mathcal{A} = \{H_t \mid t \in T\}$ . Then  $(V, \mathcal{A})$  is a hyperplane arrangement and  $W$  acts on the complement  $M = V \setminus \cup_{t \in T} H_t$ .

The action of  $W$  on  $M$  determines a representation of  $W$  on the singular cohomology of  $M$ . For  $p \geq 0$  let  $H^p(M)$  denote the  $p^{\mathrm{th}}$  singular cohomology space of  $M$  with complex coefficients and let  $H^\bullet(M) = \bigoplus_{p \geq 0} H^p(M)$  denote the total cohomology of  $M$ . It follows from a result of Brieskorn [4] that  $\dim H^\bullet(M) = |W|$  and so a naive guess would be that the representation of  $W$  on  $H^\bullet(M)$  is the regular representation of  $W$ . A simple computation for the symmetric group  $S_3$  shows that this is not the case.

In 1986, Lehrer [13] computed the character of the representation of  $W$  on  $H^\bullet(M)$  when  $W$  is a symmetric group. Subsequently, the character of  $W$  on  $H^\bullet(M)$  was computed case-by-case for other types of Coxeter groups by various authors. In 2001, Blair and Lehrer [3] gave a case-free computation of the character of  $H^\bullet(M)$ . Recently, Felder and Veselov [7] have found an elegant description of this character that describes precisely how it differs from the regular character of  $W$ . Specifically Felder and Veselov show that the character of  $H^\bullet(M)$  is given as

$$\sum_{\sigma} (2 \mathrm{Ind}_{\langle \sigma \rangle}^G(1) - \rho),$$

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where the sum runs over a set of *special* involutions in  $W$  and  $\rho$  is the regular character of  $W$ .

In contrast, while the representation of  $W$  on  $H^\bullet(M)$  is well-understood, very little is known about the representations of  $W$  on the individual graded pieces  $H^p(M)$ . Lehrer and Solomon [14] have described these representations as sums of induced representations from linear characters of centralizers of elements in  $W$  when  $W$  is the symmetric group. They conjecture that a similar decomposition exists in general. The first author [5] extended Lehrer and Solomon's construction to hyperoctahedral groups and expressed each  $H^p(M)$  as a sum of representations induced from linear characters of subgroups. However, the subgroups appearing are not always centralizers of elements of  $W$ . At the same time, Bergeron [2] found a similar decomposition of  $H^\bullet(M)$  in the hyperoctahedral case and also defined an algebra for hyperoctahedral groups that is in a sense an intermediary between  $H^\bullet(M)$  and the group algebra.

In this paper we state a conjecture that refines the conjecture of Lehrer and Solomon [14, Conjecture 1.6] and directly relates the representation of  $W$  on  $H^p(M)$  and a subrepresentation of the right regular representation of  $W$ . We prove the conjecture when  $W$  is a symmetric group. The conjecture is true for all rank two Coxeter groups and has been checked using the computer algebra system GAP [19] and package CHEVIE [8] for some low rank Coxeter groups.

The subrepresentations of the right regular representation we consider arise from the decomposition of a subalgebra of the group algebra of  $W$ , known as "Solomon's descent algebra," into projective, indecomposable modules. Projective, indecomposable modules in an artinian  $\mathbb{C}$ -algebra are generated by idempotents. These idempotents were first described explicitly by Bergeron, Bergeron, Howlett, and Taylor [1]. The second author [18] has subsequently used these idempotents to give quiver presentations for Coxeter groups. In §5.2 we show that when  $W$  is a symmetric group the subrepresentations of the right regular representation generated by these idempotents are induced representations. Schocker [20] has proved a similar result using different methods.

In section §2 of this paper we state our conjecture, in section §3 we prove some preliminary general results, and in §4 and §5 we prove the conjecture when  $W$  is a symmetric group.

## 2. THE ORLIK-SOLOMON ALGEBRA AND SOLOMON'S DESCENT ALGEBRA

We assume that  $W$  is a subgroup of the unitary group of  $V$  with respect to a fixed positive, definite, Hermitian form  $\langle \cdot, \cdot \rangle$ . It is known that  $V$  has a basis  $\Pi = \{\alpha_s \mid s \in S\}$  indexed by  $S$  so that  $\langle \alpha_s, \alpha_t \rangle = -\cos(\pi/m_{s,t})$  for  $s$  and  $t$  in  $S$ . Then  $s$  acts on  $V$  as the reflection through the hyperplane orthogonal to  $\alpha_s$  and  $\Phi = \{w(\alpha_s) \mid w \in W, s \in S\}$  is a root system in  $V$ .

**2.1. Shapes and conjugacy classes.** We begin by recalling a parameterization of the conjugacy classes in  $W$  due to Geck and Pfeiffer (see [9, §3.2]) in a form compatible with the arrangement  $(V, \mathcal{A})$  of  $W$ .

The *lattice of  $\mathcal{A}$* , denoted by  $L(\mathcal{A})$ , is the set of subspaces of  $V$  that arise as intersections of hyperplanes in  $\mathcal{A}$ :

$$L(\mathcal{A}) = \{ H_{t_1} \cap H_{t_2} \cap \cdots \cap H_{t_p} \mid t_1, t_2, \dots, t_p \in T \}.$$

For  $X$  in  $L(\mathcal{A})$  define

$$W_X = \{ w \in W \mid X \subseteq \text{Fix}(w) \}$$

to be the pointwise stabilizer of  $X$  in  $W$ . It follows from Steinberg's Theorem [24] that  $W_X$  is generated by  $\{ t \in T \mid X \subseteq H_t \}$ . It then follows that  $X = \text{Fix}(W_X)$ , and so the rule  $X \mapsto W_X$  defines an injection from  $L(\mathcal{A})$  to the set of subgroups of  $W$ . Notice that  $W_X$  is again a Coxeter group.

The action of  $W$  on  $\mathcal{A}$  induces an action of  $W$  on  $L(\mathcal{A})$ . Obviously  $wW_Xw^{-1} = W_{w(X)}$  and so for  $X$  and  $Y$  in  $L(\mathcal{A})$ , the subgroups  $W_X$  and  $W_Y$  are conjugate if and only if  $X$  and  $Y$  lie in the same  $W$ -orbit. Thus, the rule  $X \mapsto W_X$  induces a bijection between the set of orbits of  $W$  on  $L(\mathcal{A})$  and the set of conjugacy classes of subgroups  $W_X$ .

By a *shape of  $W$*  we mean a  $W$ -orbit in  $L(\mathcal{A})$ . We denote the set of shapes of  $W$  by  $\Lambda$ . For example, if  $W$  is the symmetric group  $S_n$ , then  $\Lambda$  is in bijection with the set of partitions of  $n$  and with the set of Young diagrams with  $n$  boxes.

It is shown in [17, §6.2] that the rule  $w \mapsto \text{Fix}(w)$  defines a surjection from  $W$  to  $L(\mathcal{A})$ . Composing with the map that sends an element  $X$  in  $L(\mathcal{A})$  to its  $W$ -orbit, we get a map

$$\text{sh}: W \rightarrow \Lambda.$$

We say  $\text{sh}(w)$  is the *shape of  $w$* . Thus,  $\text{sh}(w)$  is the  $W$ -orbit of  $\text{Fix}(w)$  in  $L(\mathcal{A})$ . Clearly,  $\text{sh}$  is constant on conjugacy classes and so we can define the *shape* of a conjugacy class to be the shape of any of its elements.

An element  $w$  in  $W$ , or its conjugacy class, is called *cuspidal* if  $\text{Fix}(w) = \text{Fix}(W)$ . For example, if  $W$  is the symmetric group  $S_n$ , then the conjugacy class consisting of  $n$ -cycles is the only cuspidal class. In general, there is more than one cuspidal conjugacy class. Cuspidal elements and conjugacy classes are called *elliptic* by some authors.

Suppose that  $\lambda$  is a shape,  $X$  in  $L(\mathcal{A})$  has shape  $\lambda$ , and  $C$  is a conjugacy class in  $W$  with shape  $\lambda$ . If  $w$  is in  $C$ , then  $\text{Fix}(w)$  is in the  $W$ -orbit of  $X$  and so  $C \cap W_X$  is a non-empty union of cuspidal  $W_X$ -conjugacy classes. Geck and Pfeiffer [9, §3.2] have shown that in fact  $C \cap W_X$  is a single  $W_X$ -conjugacy class. It follows that  $C \mapsto C \cap W_X$  defines a bijection between the set of conjugacy classes in  $W$  with shape  $\lambda$  and the set of cuspidal conjugacy classes in  $W_X$ .

Fix a set  $\{ X_\lambda \mid \lambda \in \Lambda \}$  of  $W$ -orbit representatives in  $L(\mathcal{A})$ . Summarizing the preceding discussion we see that conjugacy classes in  $W$  are parametrized by pairs  $(\lambda, C_\lambda)$  where  $\lambda$  is a shape and  $C_\lambda$  is a cuspidal conjugacy class in  $W_{X_\lambda}$ .

**2.2. The Orlik-Solomon algebra.** Next we consider the cohomology ring  $H^\bullet(M)$ . Arnold and Brieskorn (see [17, §1.1]) have computed  $H^\bullet(M)$ . In the following we use the presentation of this algebra given by Orlik and Solomon [15].

Recall that the set  $T$  of reflections in  $W$  parametrizes the hyperplanes in  $\mathcal{A}$ . The *Orlik-Solomon algebra of  $W$*  is the  $\mathbb{C}$ -algebra,  $A = A(\mathcal{A})$ , with generators  $\{a_t \mid t \in T\}$  and relations

- $a_{t_1}a_{t_2} = -a_{t_2}a_{t_1}$  for  $t_1$  and  $t_2$  in  $T$  and
- $\sum_{i=1}^p (-1)^i a_{t_1} \cdots \widehat{a_{t_i}} \cdots a_{t_p} = 0$  for every linearly dependent subset  $\{H_{t_1}, \dots, H_{t_p}\}$  of  $\mathcal{A}$ .

The algebra  $A$  is a skew-commutative, graded, connected  $\mathbb{C}$ -algebra that is isomorphic as a graded algebra to  $H^\bullet(M)$ . Let  $A^p$  denote the degree  $p$  subspace of  $A$ . Then

- $A^0 \cong \mathbb{C}$ ,
- for  $1 \leq p \leq |S|$  the subspace  $A^p$  is spanned by the set of all  $a_{t_1} \cdots a_{t_p}$  where  $\text{codim } H_{t_1} \cap \cdots \cap H_{t_p} = p$ , and
- $A^p = 0$  for  $p > |S|$ .

See [17, §3.1] for details.

The action of  $W$  on  $\mathcal{A}$  extends to an action of  $W$  on  $A$  as algebra automorphisms. An element  $w$  in  $W$  acts on a generator  $a_t$  of  $A$  by  $wa_t = a_{wtw^{-1}}$ . With this  $W$ -action  $A$  is isomorphic to  $H^\bullet(M)$  as graded  $W$ -algebras.

Orlik and Solomon [16] have shown that the normalizer of  $W_X$  in  $W$  is the setwise stabilizer of  $X$  in  $W$ , that is

$$N_W(W_X) = \{w \in W \mid w(X) = X\}.$$

For  $X$  in  $L(\mathcal{A})$ , define  $A_X$  to be the span of  $\{a_{t_1} \cdots a_{t_r} \mid H_{t_1} \cap \cdots \cap H_{t_r} = X\}$ . Proofs of the following statements may be found in [17, Chapter 6].

- If  $\text{codim } X = p$ , then  $A_X \subseteq A^p$ .
- There are vector space decompositions  $A^p \cong \bigoplus_{\text{codim } X=p} A_X$  and  $A \cong \bigoplus_{X \in L(\mathcal{A})} A_X$ .
- For  $w$  in  $W$ ,  $wA_X = A_{w(X)}$ . Thus,  $A_X$  is an  $N_W(W_X)$ -stable subspace of  $A$ .

For a shape  $\lambda$  in  $\Lambda$ , set  $A_\lambda = \bigoplus_{X \in \lambda} A_X$ . Suppose  $X$  is a fixed subspace in  $\lambda$  and that  $\text{codim } X = p$ . Then  $A_\lambda$  is a  $W$ -stable subspace of  $A^p$  and as representations of  $W$  we have

$$A_\lambda \cong \text{Ind}_{N_W(W_X)}^W(A_X) \quad \text{and} \quad A \cong \bigoplus_{\lambda \in \Lambda} A_\lambda$$

(see [14]).

**2.3. Solomon's descent algebra.** In contrast with the Orlik-Solomon algebra  $A$ , which is constructed from the set  $T$  of reflections in  $W$ , Solomon's descent algebra is constructed from the Coxeter generating set  $S$  of  $W$ .

Suppose that  $I$  is a subset of  $S$ . Define

$$X_I = \bigcap_{s \in I} H_s \quad \text{and} \quad W_I = \langle I \rangle.$$

Then  $X_I$  is in  $L(\mathcal{A})$  and  $\text{codim } X_I = |I|$ . It follows from Steinberg's Theorem [24] that  $W_I = W_{X_I}$ . Recall that  $\Pi = \{\alpha_s \mid s \in S\}$  is a basis of  $V$ . For  $I \subseteq S$  define

$$\Pi_I = \{\alpha_s \mid s \in I\}.$$

Then  $X_I$  is the orthogonal complement of the span of  $\Pi_I$ .

Orlik and Solomon (see [17, §6.2]) have shown that each orbit of  $W$  on  $L(\mathcal{A})$  contains a subspace  $X_I$  for some subset  $I$  of  $S$ . For subsets  $I$  and  $J$  of  $S$  define  $I \sim J$  if there is a  $w$  in  $W$  with  $w(\Pi_I) = \Pi_J$ . Then  $\sim$  is an equivalence relation. It is well-known that  $W_I$  and  $W_J$  are conjugate if and only if  $I \sim J$  (see [22]). It follows that the rule  $I \mapsto X_I$  induces a bijection between  $S/\sim$ , the set of  $\sim$ -equivalence classes, and  $\Lambda$ , the set of shapes of  $W$ .

Next, let  $\ell$  denote the length function of  $W$  determined by the generating set  $S$  and define

$$W^I = \{w \in W \mid \ell(ws) > \ell(w) \forall s \in I\}.$$

Then  $W^I$  is a set of left coset representatives of  $W_I$  in  $W$ . Also, define

$$x_I = \sum_{w \in W^I} w$$

in the group algebra  $\mathbb{C}W$ . Solomon [21] has shown that the span of  $\{x_I \mid I \subseteq S\}$  is in fact a subalgebra of  $\mathbb{C}W$ . We denote this subalgebra by  $\Sigma(W)$  and call it the *descent algebra* of  $W$ . It is not hard to see that  $\{x_I \mid I \subseteq S\}$  is linearly independent and so  $\dim \Sigma(W) = 2^{|S|}$ . Notice that  $x_S = 1$  is the identity in both  $\mathbb{C}W$  and its subalgebra  $\Sigma(W)$ .

Bergeron, Bergeron, Howlett, and Taylor [1] have defined a basis  $\{e_I \mid I \subseteq S\}$  of  $\Sigma(W)$  that consists of quasi-idempotents and is compatible with the set of shapes of  $W$ . For  $\lambda$  in  $\Lambda$  define

$$S_\lambda = \{I \subseteq S \mid X_I \in \lambda\} \quad \text{and} \quad e_\lambda = \sum_{I \in S_\lambda} e_I.$$

Then each  $e_\lambda$  is idempotent and  $\{e_\lambda \mid \lambda \in \Lambda\}$  is a complete set of primitive, orthogonal idempotents in  $\Sigma(W)$ . (See §3 for more details.) In particular,  $\sum_{\lambda \in \Lambda} e_\lambda = 1$  in  $\mathbb{C}W$ .

Define  $E_\lambda = e_\lambda \mathbb{C}W$ . In §3 we show that  $e_I \mathbb{C}W_I$  affords an action of  $N_W(W_I)$  and that  $E_\lambda$  is induced from  $e_I \mathbb{C}W_I$  when  $I$  is in  $S_\lambda$ . Thus, in analogy with the decomposition in §2.2 of the Orlik-Solomon algebra  $A$ , as representations of  $W$  we have

$$E_\lambda \cong \text{Ind}_{N_W(W_I)}^W(e_I \mathbb{C}W_I) \quad \text{and} \quad \mathbb{C}W = \bigoplus_{\lambda \in \Lambda} E_\lambda.$$

**2.4. Centralizers and complements.** The last ingredient we need in order to state the conjecture is a certain set of characters of centralizers of elements of  $W$ . These characters, together with the sign character, should quantify the difference between the representation of  $W$  on  $H^p(M)$  and a subrepresentation of the regular representation. They naturally arise in work of Howlett and Lehrer [11] and in recent results of the second author [12] that describe the structure of the centralizer of an element in  $W$ .

Suppose that  $I$  is a subset of  $S$  and  $C$  is a conjugacy class in  $W$  such that  $C \cap W_I$  is a cuspidal conjugacy class in  $W_I$ . Howlett [10] has shown that  $W_I$  has a complement,  $N_I$ , in

$N_W(W_I)$ . Moreover, it is shown in [12] that if  $c$  is in  $C \cap W_I$ , then  $Z_W(c) \subseteq N_W(W_I)$  and  $Z_{W_I}(c)W_I = N_W(W_I)$ . It follows that

(1) for any  $X$  in  $L(\mathcal{A})$ ,  $W_X$  has a complement, say  $N_X$ , in  $N_W(W_X)$  such that

$$N_W(W_X) \cong W_X \rtimes N_X,$$

and

(2) for a cuspidal element  $c$  in  $W_X$ ,  $Z_W(c) \subseteq N_W(W_X)$  and  $Z_W(c)/Z_{W_X}(c) \cong N_X$ .

Recall that  $N_W(W_X)$  is the setwise stabilizer of  $X$  in  $W$ . We may define a linear character  $\alpha_X: N_W(W_X) \rightarrow \mathbb{C}$  by  $\alpha_X(n) = \det n|_X$  for  $n$  in  $N_W(W_X)$ . For  $c$  in  $W$ , we define  $\alpha_c$  to be the restriction of  $\alpha_X$  to  $Z_W(c)$  when  $X = \text{Fix}(c)$ .

**2.5. Relating the Orlik-Solomon algebra and the descent algebra.** We now have all the concepts we need in order to state the conjecture.

Let  $\epsilon$  denote the sign character of  $W$ . For  $c$  in  $W$ , define  $X_c = \text{Fix}(c)$ ,  $\text{rk}(c) = \text{codim } X_c$ , and  $W_c = W_{X_c}$ . Notice that  $c$  is a cuspidal element in  $W_c$ .

Associated with each  $\lambda$  in  $\Lambda$  we have the  $W$ -stable subspace  $A_\lambda$  of the Orlik-Solomon algebra  $A$ , a right ideal  $E_\lambda$  in  $\mathbb{C}W$ , and the set of conjugacy classes with shape  $\lambda$ . We conjecture that  $A_\lambda$  and  $E_\lambda$  are related to the set of conjugacy classes with shape  $\lambda$  as follows.

**Conjecture 2.1.** *Choose a set  $\mathcal{C}$  of conjugacy class representatives in  $W$ . For  $\lambda$  in  $\Lambda$  set  $\mathcal{C}_\lambda = \{c \in \mathcal{C} \mid \text{sh}(c) = \lambda\}$ . Then, for each conjugacy class representative  $c$  in  $\mathcal{C}$ , there is a linear character  $\phi_c$  of  $Z_W(c)$ , such that for every  $\lambda$  in  $\Lambda$ ,*

(a) *the character of  $E_\lambda$  is*

$$\bigoplus_{c \in \mathcal{C}_\lambda} \text{Ind}_{Z_W(c)}^W(\phi_c) = \bigoplus_{c \in \mathcal{C}_\lambda} \text{Ind}_{N_W(W_c)}^W(\text{Ind}_{Z_W(c)}^{N_W(W_c)}(\phi_c))$$

and

(b) *the character of  $A_\lambda$  is*

$$\bigoplus_{c \in \mathcal{C}_\lambda} \text{Ind}_{Z_W(c)}^W(\epsilon_c \alpha_c \phi_c) = \epsilon \bigoplus_{c \in \mathcal{C}_\lambda} \text{Ind}_{N_W(W_c)}^W(\alpha_{X_c} \text{Ind}_{Z_W(c)}^{N_W(W_c)}(\phi_c)),$$

where  $\epsilon_c$  denotes the restriction of  $\epsilon$  to  $Z_W(c)$ .

In particular,

$$H^p(M) \cong A^p \cong \bigoplus_{\text{rk}(c)=p} \text{Ind}_{Z_W(c)}^W(\epsilon_c \alpha_c \phi_c)$$

for  $0 \leq p \leq |S|$ , and

$$\mathbb{C}W \cong \bigoplus_{c \in \mathcal{C}} \text{Ind}_{Z_W(c)}^W(\phi_c) \quad \text{and} \quad H^\bullet(M) \cong \bigoplus_{c \in \mathcal{C}} \text{Ind}_{Z_W(c)}^W(\epsilon_c \alpha_c \phi_c).$$

We show in Corollary 3.5 that  $\dim A_\lambda$  is the number of elements in  $W$  with shape  $\lambda$ . Bergeron, Bergeron, Howlett, and Taylor [1] have shown that  $\dim E_\lambda$  is also the number of elements in  $W$  with shape  $\lambda$ . Thus  $\dim A_\lambda = \dim E_\lambda$ .

As stated in the introduction, the main result in this paper is a proof of Conjecture 2.1 for symmetric groups. The conjecture is known to be true for all Coxeter groups with rank at most two [6].

We in fact prove a slightly stronger result than is stated in the conjecture. We show that the character  $\phi_c$  of  $Z_W(c)$  may be chosen to be a character of the normal subgroup  $Z_{W_c}(c)$  of  $Z_W(c)$  that is  $N_{X_c}$ -stable and extends to a character of  $Z_W(c)$ .

### 3. $E_\lambda$ IS AN INDUCED REPRESENTATION

Suppose  $\lambda$  is in  $\Lambda$ . We have seen in §2.2 that  $A_\lambda \cong \text{Ind}_{N_W(W_X)}^W(A_X)$  for  $X$  in  $\lambda$ . In this section we show that  $E_\lambda$  has a similar description as an induced representation for  $I$  in  $S_\lambda$ . We begin by recalling some results of Bergeron, Bergeron, Howlett, and Taylor [1].

For subsets  $I, J$ , and  $K$  of  $S$  define

$$W^{IJ} = (W^I)^{-1} \cap W^J \quad \text{and} \quad W^{IJK} = \{w \in W^{IJ} \mid w^{-1}(\Pi_I) \cap \Pi_J = \Pi_K\}.$$

Then  $W^{IJ}$  is the set of minimal length  $(W_I, W_J)$ -double coset representatives in  $W$ . Solomon [21] has shown that  $x_I x_J = \sum_K a_{IJK} x_K$  where  $a_{IJK} = |W^{IJK}|$ .

The quasi-idempotents  $e_I$  in §2.3 are defined as follows. For subsets  $J$  and  $K$  of  $S$  set

$$m_{JK} = \left| \coprod_{L \sim J} \{w \in W^{LK} \mid w^{-1}(\Pi_L) \cap \Pi_K = \Pi_J\} \right|.$$

Then  $m_{JK} = 0$  when  $J \not\subseteq K$  and  $m_{JK} = |\{w \in W^K \mid w(\Pi_J) \subseteq \Pi\}|$  when  $J \subseteq K$ . Moreover,  $m_{JJ} \neq 0$  for all  $J$  and so the system of equations

$$x_K = \sum_{J \subseteq S} m_{JK} e_J, \quad K \subseteq S$$

can be solved for  $\{e_J \mid J \subseteq S\}$ . Define  $n_{JK}$  and  $e_K$  by

$$e_K = \sum_{J \subseteq S} n_{JK} x_J.$$

Notice that  $n_{JJ} = m_{JJ}^{-1}$  and  $n_{JK} = 0$  when  $J \not\subseteq K$ .

The next two lemmas give some translation properties for the quantities just defined.

**Lemma 3.1.** *Suppose that  $K \subseteq S$  and  $d$  is in  $W$  with  $d^{-1}(\Pi_K) \subseteq \Pi$ .*

- (a)  $W^K d = W^{K^d}$ .
- (b)  $x_K d = x_{K^d}$ .
- (c)  $m_{I^d J^d} = m_{IJ}$  for  $I \subseteq J \subseteq K$ .
- (d)  $e_L d = e_{L^d}$  for  $L \subseteq K$ .

*Proof.* Statement (a) is proved in [1, Lemma 2.4]. Statement (b) follows immediately from (a).

Suppose that  $I \subseteq J \subseteq K$ . Clearly,  $\Pi_{L^d} = d^{-1}(\Pi_L)$  for all  $L \subseteq K$  and so

$$\begin{aligned} m_{I^d J^d} &= |\{w \in W^{J^d} \mid w(\Pi_{I^d}) \subseteq \Pi\}| \\ &= |\{w \in W^{J^d} \mid wd^{-1}(\Pi_I) \subseteq \Pi\}| \\ &= |\{wd^{-1} \in W^J \mid wd^{-1}(\Pi_I) \subseteq \Pi\}| \\ &= |\{w \in W^J \mid w(\Pi_I) \subseteq \Pi\}| \\ &= m_{IJ}. \end{aligned}$$

This proves (c).

Using (b) and (c) we see that for  $J \subseteq K$ ,

$$\sum_I m_{IJ}(e_I d) = x_J d = x_{J^d} = \sum_I m_{IJ^d} e_I = \sum_I m_{dIJ} e_I = \sum_I m_{IJ} e_{I^d}.$$

Thus,  $\sum_I m_{IJ}(e_I d) = \sum_I m_{IJ} e_{I^d}$ . Now, fix a subset  $L$  of  $K$ , multiply both sides by  $n_{JL}$ , and sum over  $J$ , to get  $e_L d = e_{L^d}$ . (Note that  $n_{JL} = 0$  unless  $J \subseteq L$ .) This proves (d).  $\square$

**Lemma 3.2.** *Suppose  $n$  is in  $\text{GL}(V)$  with  $n(\Pi) = \Pi$ . Then  $n^{-1}e_I n = e_{I^n}$  for  $I \subseteq S$ . In particular,  $n$  centralizes  $e_S$  in  $\mathbb{C}W$ .*

*Proof.* It follows from the assumption that  $n(\Pi) = \Pi$  that  $\ell(nwn^{-1}) = \ell(w)$  for all  $w$  in  $W$ . Therefore,  $n^{-1}W^I n = W^{I^n}$  and hence  $n^{-1}x_I n = x_{I^n}$  for all  $I \subseteq S$ .

Suppose  $I \subseteq J \subseteq S$ . Then

$$\begin{aligned} m_{I^n J^n} &= |\{w \in W^{J^n} \mid w(\Pi_{I^n}) \subseteq \Pi\}| \\ &= |\{w \in n^{-1}W^J n \mid wn^{-1}(\Pi_I) \subseteq \Pi\}| \\ &= |\{nwn^{-1} \in W^J \mid nwn^{-1}(\Pi_I) \subseteq n(\Pi)\}| \\ &= |\{w \in W^J \mid w(\Pi_I) \subseteq \Pi\}| \\ &= m_{IJ}. \end{aligned}$$

Thus, we see that for  $K \subseteq S$ ,

$$\sum_J m_{JK}(n^{-1}e_J n) = n^{-1}x_K n = x_{K^n} = \sum_J m_{JK^n} e_J = \sum_J m_{nJK} e_J = \sum_J m_{JK} e_{J^n}.$$

Therefore,  $\sum_J m_{JK}(n^{-1}e_J n) = \sum_J m_{JK} e_{J^n}$ . Now, fix a subset  $I$  of  $S$ , multiply both sides by  $n_{KI}$ , and sum over  $K$ , to get  $n^{-1}e_I n = e_{I^n}$ .  $\square$

Suppose  $\lambda$  is in  $\Lambda$ . Recall that  $e_\lambda = \sum_{I \in S_\lambda} e_I$  and  $E_\lambda = e_\lambda \mathbb{C}W$ . It is shown in [1, §7] that if  $I$  and  $J$  are in  $S_\lambda$ , then

$$e_I e_J = \frac{1}{|S_\lambda|} e_J.$$

It follows that  $e_\lambda$  is an idempotent. It is also shown in [1, §7] that  $1 = \sum_{\lambda \in \Lambda} e_\lambda$  and that  $\{e_\lambda \mid \lambda \in \Lambda\}$  is a complete set of pairwise orthogonal, primitive idempotents in  $\Sigma(W)$ . Since



$\Sigma(W)$  is a subalgebra of  $\mathbb{C}W$ , it follows that  $\{e_\lambda \mid \lambda \in \Lambda\}$  is a set of pairwise orthogonal idempotents in  $\mathbb{C}W$ .

**Lemma 3.3.** *Suppose that  $\lambda$  is in  $\Lambda$  and  $I$  is in  $S_\lambda$ . Then,  $E_\lambda = e_I \mathbb{C}W$ .*

*Proof.* Say  $S_\lambda = \{J_1, \dots, J_r\}$ . For  $1 \leq i \leq r$ , choose  $d_i$  in  $W$  with  $J_i = I^{d_i}$ . Then  $e_{J_i} = e_I d_i$  by Lemma 3.1(d). Therefore,  $e_\lambda = e_I(d_1 + \dots + d_r)$  and so  $E_\lambda \subseteq e_I \mathbb{C}W$ . On the other hand,

$$e_\lambda e_I = (e_{J_1} + \dots + e_{J_r})e_I = e_{J_1}e_I + \dots + e_{J_r}e_I = \frac{1}{r}e_I + \dots + \frac{1}{r}e_I = e_I,$$

and so  $e_I \mathbb{C}W = e_\lambda e_I \mathbb{C}W \subseteq e_\lambda \mathbb{C}W$ .  $\square$

We next compute the dimensions of the various spaces we are studying in terms of shapes.

**Lemma 3.4.** *Suppose that  $\lambda$  is a shape in  $\Lambda$ ,  $X$  is a subspace in  $\lambda$ , and  $C$  is a conjugacy class in  $W$  with shape  $\lambda$ . Then*

- (a)  $|C| = |W : N_W(W_X)| |C \cap W_X|$  and
- (b)  $|\text{sh}^{-1}(\lambda)| = |W : N_W(W_X)| |\text{sh}^{-1}(\lambda) \cap W_X|$ .

*Proof.* Notice that  $C \cap W_X$  is a cuspidal conjugacy class in  $W_X$ . Thus, it follows from (1) and (2) in §2.4 that  $|N_W(W_X) : Z_W(c)| = |W_X : Z_{W_X}(c)|$  for  $c$  in  $C$ . Therefore

$$|C| = |W : N_W(W_X)| |N_W(W_X) : Z_W(c)| = |W : N_W(W_X)| |C \cap W_X|.$$

This proves (a). Statement (b) follows from (a) and the observation that  $\text{sh}^{-1}(\lambda)$  is the union of those conjugacy classes in  $W$  whose intersection with  $W_X$  is a cuspidal conjugacy class in  $W_X$ .  $\square$

The quasi-idempotents  $e_I$  are defined relative to the ambient set  $S$ . We use a superscript to indicate this ambient set when it is not equal  $S$ . Thus, for  $I \subseteq J \subseteq S$ ,  $e_I^J$  denotes the quasi-idempotent in  $\mathbb{C}W_J$  defined using  $J$  as the ambient set instead of  $S$ .

**Corollary 3.5.** *Suppose that  $\lambda$  is in  $\Lambda$  and  $I$  is in  $S_\lambda$ . Then*

- (a)  $\dim A_{X_I} = \dim e_I \mathbb{C}W_I = |\text{sh}^{-1}(\lambda) \cap W_I|$  and
- (b)  $\dim A_\lambda = \dim E_\lambda = |\text{sh}^{-1}(\lambda)|$ .

*Proof.* It is clear that  $\{w \in W \mid \text{Fix}(w) = X_I\} = \text{sh}^{-1}(\lambda) \cap W_I$  is the set of all cuspidal elements in  $W_I$ . It is shown in [5, Proposition 2.4] that  $\dim A_{X_I} = |\{w \in W \mid \text{Fix}(w) = X_I\}|$ . Therefore,  $\dim A_{X_I} = |\text{sh}^{-1}(\lambda) \cap W_I|$ , and it follows from Lemma 3.4(b) that  $\dim A_\lambda = |\text{sh}^{-1}(\lambda)|$ . As remarked above, Bergeron, Bergeron, Howlett, and Taylor [1] have shown that  $\dim E_\lambda = |\text{sh}^{-1}(\lambda)|$ . This proves (b).

To complete the proof of (a) it remains to show that  $\dim e_I \mathbb{C}W_I = |\text{sh}^{-1}(\lambda) \cap W_I|$ . It is shown in [1, Proposition 7.3] that  $e_I$  factors as  $e_I = x_I e_I^I$ . Thus,  $e_I \mathbb{C}W_I = x_I e_I^I \mathbb{C}W_I$ . Also, it follows from [1, Theorem 7.15] that  $\dim e_I^I \mathbb{C}W_I = |\text{sh}^{-1}(\lambda) \cap W_I|$ . Because  $W^I$  is a complete set of left coset representatives for  $W_I$  in  $W$ , it is clear that left multiplication by  $x_I$  defines an isomorphism of right  $\mathbb{C}W_I$ -modules  $e_I^I \mathbb{C}W_I \cong e_I \mathbb{C}W_I$ . Therefore,  $\dim e_I \mathbb{C}W_I = |\text{sh}^{-1}(\lambda) \cap W_I|$  as desired.  $\square$

We can now show that  $E_\lambda$  is an induced representation.

**Proposition 3.6.** *Suppose that  $\lambda$  is in  $\Lambda$  and  $I$  is in  $S_\lambda$ .*

(a)  $N_W(W_I)$  acts on  $e_I\mathbb{C}W_I$  by right multiplication and  $E_\lambda \cong \text{Ind}_{N_W(W_I)}^W(e_I\mathbb{C}W_I)$ .

(b)  $N_W(W_X)$  acts on  $A_X$  and  $A_\lambda \cong \text{Ind}_{N_W(W_X)}^W(A_X)$ .

*Proof.* Statement (b) is proved in [14, §2]. We prove (a).

Recall that  $N_W(W_I) = W_I N_I$ . Obviously  $e_I\mathbb{C}W_I$  is stable under right multiplication by  $W_I$ . We have seen in the proof of Corollary 3.5 that  $e_I\mathbb{C}W_I = x_i e_I^I \mathbb{C}W_I$ . If  $n$  is in  $N_I$ , then  $n(\Pi_I) = \Pi_I$  and it follows from Lemmas 3.1 and 3.2 that

$$e_I\mathbb{C}W_I n = e_I n \mathbb{C}W_I = (x_I n)(n^{-1} e_I^I n) \mathbb{C}W_I = x_I e_I^I \mathbb{C}W_I = e_I \mathbb{C}W_I.$$

Therefore,  $e_I\mathbb{C}W_I$  is stable under right multiplication by  $N_W(W_I)$ .

Since  $E_\lambda = e_I\mathbb{C}W$ , by Lemma 3.3, to prove that  $E_\lambda \cong \text{Ind}_{N_W(W_I)}^W(e_I\mathbb{C}W_I)$  it is enough to show that the multiplication map  $e_I\mathbb{C}W_I \otimes_{\mathbb{C}N_W(W_I)} \mathbb{C}W \rightarrow E_\lambda$  is a bijection. The map is obviously a surjection. Moreover, using Lemma 3.4 and Corollary 3.5, we have

$$\begin{aligned} \dim E_\lambda &= |\text{sh}^{-1}(\lambda)| \\ &= |W : N_W(W_I)| |\text{sh}^{-1}(\lambda) \cap W_I| \\ &= |W : N_W(W_I)| \dim e_I\mathbb{C}W_I \\ &= \dim e_I\mathbb{C}W_I \otimes_{\mathbb{C}N_W(W_I)} \mathbb{C}W \end{aligned}$$

and so the multiplication map is a bijection. □

#### 4. SYMMETRIC GROUPS: $\lambda = (n)$

The rest of this paper is devoted to the proof of Conjecture 2.1 for symmetric groups.

From now on, we take  $W$  to be the symmetric group on  $n$  letters with  $n \geq 2$  and we identify  $W$  with the subgroup of  $\text{GL}_n(\mathbb{C})$  that acts on the basis  $\{v_1, v_2, \dots, v_n\}$  as permutations. Here,  $v_i$  is the column vector whose  $j^{\text{th}}$  entry is 0 for  $j \neq i$  and 1 for  $j = i$ . For  $1 \leq i \leq n-1$  let  $s_i$  denote the matrix in  $W$  that interchanges  $v_i$  and  $v_{i+1}$  and fixes  $v_j$  for  $j \neq i, i+1$ . Then  $S = \{s_1, s_2, \dots, s_{n-1}\}$  is a Coxeter generating set for  $W$ .

By a *partition of  $n$*  we mean a non-increasing finite sequence of positive integers whose sum is  $n$ . Say  $\lambda = (\lambda_1, \dots, \lambda_p)$  is a partition of  $n$ . Then  $\lambda_1 \geq \dots \geq \lambda_p > 0$  and  $\sum_{k=1}^p \lambda_k = n$ . The integers  $\lambda_i$  are called the *parts of  $\lambda$* .

It is well-known that for  $W = S_n$  we may identify  $\Lambda$  with the set of partitions of  $n$ . We make this identification precise as follows. Suppose that  $\lambda$  is a partition of  $n$  with  $p$  parts. Define partial sums  $\tau_i$  for  $i = 0, 1, \dots, p$  by  $\tau_0 = 0$  and  $\tau_i = \lambda_1 + \dots + \lambda_i$  for  $1 \leq i \leq p$ . Define

$$I_\lambda = S \setminus \{s_{\tau_1}, s_{\tau_2}, \dots, s_{\tau_{p-1}}\} \quad \text{and} \quad W_\lambda = \langle I_\lambda \rangle.$$

Then  $W_\lambda$  is isomorphic to the product of symmetric groups  $S_{\lambda_1} \times \dots \times S_{\lambda_p}$ , where the factor  $S_{\lambda_i}$  acts on the subset  $\{v_{\tau_{i-1}+1}, v_{\tau_{i-1}+2}, \dots, v_{\tau_i}\}$  of  $\{v_1, v_2, \dots, v_n\}$ . Next, define

$$X_\lambda = \text{Fix}(W_\lambda).$$

Then  $X_\lambda$  is in  $L(\mathcal{A})$  and  $W_{X_\lambda} = W_\lambda$ . We have seen in Proposition 3.6 that

$$E_\lambda \cong \text{Ind}_{N_W(W_\lambda)}^W(e_{I_\lambda} \mathbb{C}W_\lambda) \quad \text{and} \quad A_\lambda \cong \text{Ind}_{N_W(W_\lambda)}^W(A_{X_\lambda}).$$

It is well-known and straightforward to check that  $\{X_\lambda \mid \lambda \text{ is a partition of } n\}$  is a complete set of orbit representatives for the action of  $W$  on  $L(\mathcal{A})$  and that  $\{I_\lambda \mid \lambda \text{ is a partition of } n\}$  is a complete set of representatives for  $S/\sim$ .

Notice that in the extreme case when all parts of  $\lambda$  are equal 1 we have  $I_\lambda = \emptyset$  and  $W_\lambda = W_\emptyset = \{1\}$ . At the other extreme, when  $\lambda = (n)$ , we have  $I_\lambda = S$  and  $W_\lambda = W_S = W$ . We first prove Conjecture 2.1 when  $\lambda = (n)$ .

For the rest of this section we take  $\lambda = (n)$ . Then  $W_\lambda = N_W(W_\lambda) = W$  and so  $E_{(n)} = e_{I_{(n)}} \mathbb{C}W_{(n)}$  and  $A_{(n)} = A_{X_{(n)}}$ . Moreover,  $A_{X_{(n)}} = A^{n-1}$  is the top, non-zero graded piece of  $A$ . To simplify the notation, we denote  $A_{(n)}$ ,  $E_{(n)}$ , and  $e_{I_{(n)}}$  by  $A_n$ ,  $E_n$ , and  $e_n$  respectively.

Define  $c_1 = 1$  in  $W$  and for  $1 \leq i \leq n$  define  $c_i = s_{i-1} \cdots s_2 s_1$ , so  $c_i$  acts on the basis  $\{v_1, v_2, \dots, v_n\}$  as an  $i$ -cycle. Also, set  $c = c_n$ . Then,

- $c$  is a cuspidal element in  $W$ ,
- the set of cuspidal elements in  $W$  is precisely the conjugacy class of  $c$ , and
- $Z_W(c) = \langle c \rangle$  is the cyclic group of order  $n$  generated by  $c$ .

Set  $\zeta = e^{2\pi i/n}$  in  $\mathbb{C}$  and define  $\phi: Z_W(c) \rightarrow \mathbb{C}$  by  $\phi(c^{-1}) = \zeta$ . The elements we have denoted by  $c_i$  are denoted by  $c_i^{-1}$  by Lehrer and Solomon [14]. However, the character  $\phi$  of  $Z_W(c)$  is the same as in [14].

**Theorem 4.1.** *With the preceding notation we have that*

- (a) *the character of  $W$  on  $E_n$  is  $\text{Ind}_{Z_W(c)}^W(\phi)$  and*
- (b) *the character of  $W$  on  $A_n$  is  $\epsilon \text{Ind}_{Z_W(c)}^W(\phi)$ .*

Notice that with the notation of §2.1, we have  $Z_c = Z_W(c)$  and so  $\phi = \phi_c$ .

Statement (b) has been proved by Stanley [23, Theorem 7.2] and by Lehrer and Solomon [14, Theorem 3.9]. Our proof below that the character of  $W$  on  $E_n$  is  $\text{Ind}_{Z_W(c)}^W(\phi)$  follows the Lehrer-Solomon argument. To emphasize and differentiate the parallel arguments, we use the convention that the superscript  $+$  denotes quantities associated with  $E_n$  and the superscript  $-$  denotes quantities associated with  $A_n$ .

Suppose  $t$  is an indeterminate. For  $0 \leq k \leq n$ , define elements  $b^+(n, k)$  and  $b^-(n, k)$  in  $\mathbb{C}W$  by

$$(1 - c_1 t)(1 - c_2 t) \cdots (1 - c_n t) = \sum_{k=0}^n b^+(n, k) t^k$$

and

$$(1 + c_n t)(1 - c_{n-1} t) \cdots (1 + (-1)^{n-1} c_1 t) = \sum_{k=0}^n b^-(n, k) t^k$$

respectively (the  $k^{\text{th}}$  factor in the product on the left-hand side of the last equation is  $(1 + (-1)^{k-1}c_{n-k+1}t)$ ).

Set  $W_{n-1} = \langle s_1, s_2, \dots, s_{n-2} \rangle$ . Then  $W_{n-1} \cong S_{n-1}$ . The analog of the idempotent  $e_n$  in  $E_n$  is the basis element  $a_n = a_{s_1}a_{s_2} \cdots a_{s_{n-1}}$  in  $A_n = A^{n-1}$ . Lehrer and Solomon [14, §3] prove the following statements.

- (i)  $A_n = \mathbb{C}W a_n$ .
- (ii)  $c^{-k}a_n = b^-(n-1, k)a_n$  for  $0 \leq k \leq n-1$ . In particular,  $A_n = \mathbb{C}W_{n-1}a_n$ .
- (iii) Consider the homomorphism of left  $\mathbb{C}W$ -modules from  $\mathbb{C}W$  to  $A_n$  given by right multiplication by  $a_n$ . The kernel of this mapping is the left  $\mathbb{C}W_{n-1}$ -module generated by  $\{c^{-k} - b^-(n-1, k) \mid 0 \leq k \leq n-1\}$ .
- (iv)  $\{wa_n \mid w \in W_{n-1}\}$  is a  $\mathbb{C}$ -basis of  $A_n$  and  $A_n$  is the left regular  $\mathbb{C}W_{n-1}$ -module.

Next we show that the analogous statements hold with  $A_n$  replaced by  $E_n$  and  $b^-(n, k)$  replaced by  $b^+(n, k)$ .

For  $k = 1, 2, \dots, n-1$ , define  $x_k = x_{S \setminus \{s_k\}}$  and  $w_k = c_1c_2 \cdots c_k$ . Then  $w_k$  is the longest element in  $\langle s_1, s_2, \dots, s_{k-1} \rangle$ .

**Lemma 4.2.** *Suppose  $1 \leq k \leq n-1$ . Then*

$$W^{S \setminus \{s_k\}} w_k = \{c_{i_1} \cdots c_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}.$$

*Proof.* It suffices to show that if  $1 \leq i_1 < \cdots < i_k \leq n$ , then  $c_{i_1} \cdots c_{i_k} w_k$  is in  $W^{S \setminus \{s_k\}}$ . For this, we consider elements in  $W$  as acting on  $\{1, \dots, n\}$ . That is, we identify the vector  $v_j$  with  $j$  for  $1 \leq j \leq n$ . Then

$$W^{S \setminus \{s_k\}} = \{w \in W \mid w(j) < w(j+1) \forall j \in \{1, \dots, n-1\} \setminus \{k\}\}$$

and

$$w_k(j) = \begin{cases} k+1-j & 1 \leq j \leq k \\ j & k+1 \leq j \leq n. \end{cases}$$

Fix  $i_1, \dots, i_k$  with  $1 \leq i_1 < \cdots < i_k \leq n$  and set  $w = c_{i_1} \cdots c_{i_k} w_k$ . If  $1 \leq j \leq k$ , then  $w(j) = i_j < i_{j+1} = w(j+1)$ . If  $j \geq i_k$ , then  $w(j) \leq j < j+1 = w(j+1)$ . Suppose that  $k < j < i_k$ . Choose  $r$  minimal such that

$$j+1 \leq i_k, j+1-1 \leq i_{k-1}, \dots, j+1-r \leq i_{k-r}, \text{ and } j+1-r-1 > i_{k-r-1}.$$

Then  $w(j) \leq j-r-1 < j-r = j+1-r-1 = w(j+1)$ . □

**Corollary 4.3.** *For  $1 \leq k \leq n-1$ , we have  $b^+(n, k) = (-1)^k x_k w_k$ .*

*Proof.* Using the definition and Lemma 4.2 we have

$$b^+(n, k) = (-1)^k \sum_{1 \leq i_1 < \cdots < i_k \leq n} c_{i_1} \cdots c_{i_k} = (-1)^k x_k w_k.$$

□

**Proposition 4.4.** *The following analogs of (i)-(iv) above hold.*

- (a)  $E_n = e_n \mathbb{C}W$ .
- (b)  $e_n c^k = e_n b^+(n-1, k)$  for  $0 \leq k \leq n-1$ . In particular,  $E_n = e_n \mathbb{C}W_{n-1}$ .
- (c) Consider the endomorphism of  $\mathbb{C}W$  considered as a right  $\mathbb{C}W$ -module given by left multiplication by  $e_n$ . The kernel of this mapping is the free, right  $\mathbb{C}W_{n-1}$ -module with basis  $\{c^k - b^+(n-1, k) \mid 0 \leq k \leq n-1\}$ .
- (d)  $\{e_n w \mid w \in W_{n-1}\}$  is a  $\mathbb{C}$ -basis of  $E_n$  and  $E_n$  is the right regular  $\mathbb{C}W_{n-1}$ -module.

*Proof.* The first statement follows immediately from the definitions.

We prove (b) by recursion. It is clear that  $e_n c^k = e_n b^+(n-1, k)$  for  $k=0$ , since  $b(n-1, 0) = 1 = c^0$ . Suppose  $e_n c^{k-1} = e_n b^+(n-1, k-1)$ . It follows from [1, Theorem 7.8] that  $e_n x_J = 0$  unless  $J = S$ . Thus, it follows from Corollary 4.3 that  $e_n b^+(n, k) = (-1)^k e_n x_k w_k = 0$  for  $1 \leq k \leq n-1$ . On the other hand, it follows from the definition that

$$\sum_{k=0}^n b^+(n, k) t^k = \left( \sum_{k=0}^{n-1} b^+(n-1, k) t^k \right) (1 - c_n t)$$

and hence  $b^+(n, k) = b^+(n-1, k) - b^+(n-1, k-1)c$  for  $1 \leq k \leq n-1$ . Therefore,

$$e_n c^k = e_n c^{k-1} c = e_n b^+(n-1, k-1) c = e_n b^+(n-1, k).$$

Next, consider the endomorphism of  $\mathbb{C}W$  given by  $x \mapsto e_n x$ . Let  $K$  denote the kernel of this mapping and let  $K_1$  denote the  $\mathbb{C}W_{n-1}$ -submodule of  $\mathbb{C}W$  generated by  $\{c^k - b^+(n-1, k) \mid 0 \leq k \leq n-1\}$ . It follows from (b) that  $K_1 \subseteq K$ . Moreover,  $\{c^k - b^+(n-1, k) \mid 0 \leq k \leq n-1\}$  is a  $\mathbb{C}W_{n-1}$  basis of  $K_1$  because the cyclic subgroup generated by  $c$  is a left transversal of  $W_{n-1}$  in  $W$ . Therefore,  $\dim_{\mathbb{C}} K_1 = (n-1)(n-1)!$ . However,

$$\dim K = \dim \mathbb{C}W - \dim E_n = n! - |W : Z_W(c)| = (n-1)(n-1)! = \dim K_1.$$

Therefore  $K_1 = K$ . This proves (c).

Because  $b^+(n-1, k)$  is in  $\mathbb{C}W_{n-1}$  for  $1 \leq k \leq n-1$ , it follows from (b) that the image of the mapping  $x \mapsto e_n x$  is  $e_n \mathbb{C}W_{n-1}$ . Therefore,  $E_n = e_n \mathbb{C}W_{n-1}$ . Since  $\dim E_n = (n-1)!$ , it follows that  $\{e_n w \mid w \in W_{n-1}\}$  is a  $\mathbb{C}$ -basis of  $E_n$ . This proves (d).  $\square$

Finally, define idempotents  $f^+$  and  $f^-$  in  $\mathbb{C}Z_W(c)$  by

$$f^+ = \frac{1}{n} \sum_{k=0}^{n-1} \phi(c^k) c^{-k} = \frac{1}{n} \sum_{k=0}^{n-1} \zeta^{-k} c^{-k}$$

and

$$f^- = \frac{1}{n} \sum_{k=0}^{n-1} \epsilon(c)^k \phi(c^k) c^{-k} = \frac{1}{n} \sum_{k=0}^{n-1} \epsilon(c)^k \zeta^{-k} c^{-k}.$$

Obviously, the lines  $\mathbb{C}f^+$  and  $\mathbb{C}f^-$  in  $\mathbb{C}W$  are stable under left and right multiplication by  $Z_W(c)$  and afford the characters  $\phi$  and  $\epsilon\phi$  of  $Z_W(c)$  respectively. Moreover,  $\text{Ind}_{Z_W(c)}^W(\phi)$  is afforded by the right  $\mathbb{C}W$ -module  $f^+ \mathbb{C}W$  and  $\epsilon \text{Ind}_{Z_W(c)}^W(\phi) = \text{Ind}_{Z_W(c)}^W(\epsilon\phi)$  is afforded by the left  $\mathbb{C}W$ -module  $\mathbb{C}W f^-$ . Thus, to prove Theorem 4.1 it is enough to find  $\mathbb{C}W$ -isomorphisms  $E_n \cong f^+ \mathbb{C}W$  and  $A_n \cong \mathbb{C}W f^-$ .

**Lemma 4.5.** *The idempotent  $f^+$  acts invertibly by right multiplication on  $e_n$  and the idempotent  $f^-$  acts invertibly by left multiplication on  $a_n$ .*

*Proof.* Lehrer and Solomon [14, §3] show that  $f^-$  acts invertibly on  $a_n$ . Their argument is easily modified to show that  $f^+$  acts invertibly by right multiplication on  $e_n$  as follows.

We have  $(1 - c_1\zeta) \cdots (1 - c_{n-1}\zeta) = \sum_{k=0}^{n-1} b^+(n-1, k)\zeta^k$ . Multiply both sides on the left by  $\frac{1}{n}e_n$  and use Proposition 4.4(b) to get

$$\frac{1}{n}e_n(1 - \zeta c_1) \cdots (1 - \zeta c_{n-1}) = \frac{1}{n} \sum_{k=0}^{n-1} \zeta^k e_n b^+(n-1, k) = \frac{1}{n} \sum_{k=0}^{n-1} \zeta^k e_n c^k = e_n f^+.$$

If  $1 \leq k \leq n-1$ , then

$$1 - \zeta^k = 1 - \zeta^k c_k^k = (1 - \zeta c_k)(1 + \zeta c_k + \cdots + \zeta^{k-1} c_k^{k-1}).$$

Since  $\zeta$  is a primitive  $n^{\text{th}}$  root of unity,  $1 - \zeta^k \neq 0$  in  $\mathbb{C}$ . Thus,  $1 - \zeta c_k$  acts invertibly on  $e_n$  for  $1 \leq k \leq n-1$  and so  $f^+$  acts invertibly on  $e_n$ .  $\square$

*Proof of Theorem 4.1.* (See [14, §3].) Consider the mapping from  $f^+\mathbb{C}W$  to  $E_n$  given by  $x \mapsto e_n x$ . It follows from Lemma 4.5 and the discussion preceding it that  $e_n f^+ \neq 0$ , that  $Z_W(c)$  acts on the line  $\mathbb{C}e_n f^+$  in  $E_n$  as the character  $\phi$ , and that the mapping is a surjection. Since  $\dim f^+\mathbb{C}W = |W : Z_W(c)| = (n-1)! = \dim E_n$ , the mapping is also an injection. Thus, we have an isomorphism of right  $\mathbb{C}W$ -modules,  $E_n \cong f^+\mathbb{C}W$ .

As in [14, §3], similar reasoning applies to the mapping from  $\mathbb{C}W f^-$  to  $A_n$  given by  $x \mapsto x a_n$  and shows that  $A_n \cong \mathbb{C}W f^-$ .  $\square$

## 5. SYMMETRIC GROUPS: ARBITRARY $\lambda$

In this section we consider the case of an arbitrary partition of  $n$  and complete the proof of Conjecture 2.1 for symmetric groups.

Suppose  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$  is a partition of  $n$ . Recall that  $I_\lambda = S \setminus \{s_{\tau_1}, s_{\tau_2}, \dots, s_{\tau_{p-1}}\}$  and that  $W_\lambda = \langle I_\lambda \rangle$  is isomorphic to the product of symmetric groups  $S_{\lambda_1} \times \cdots \times S_{\lambda_p}$ , where the factor  $S_{\lambda_i}$  acts on  $\{v_{\tau_{i-1}+1}, v_{\tau_{i-1}+2}, \dots, v_{\tau_i}\}$ . For  $1 \leq i \leq p$  define  $g_{\lambda_i} = s_{\tau_{i-1}} \cdots s_{\tau_{i-1}+2} s_{\tau_{i-1}+1}$ . Then  $g_{\lambda_i}$  is the  $\lambda_i$ -cycle in  $S_{\lambda_i}$  that corresponds to the  $n$ -cycle  $c = c_n$  in §4. Next, define  $c_\lambda = g_{\lambda_1} g_{\lambda_2} \cdots g_{\lambda_p}$  and  $Z_\lambda = Z_{W_\lambda}(c_\lambda)$ . Then

- $c_\lambda$  is a cuspidal element in  $W_\lambda$ ,
- the set of cuspidal elements in  $W_\lambda$  is precisely the conjugacy class of  $c_\lambda$ , and
- $Z_\lambda \cong \langle g_{\lambda_1} \rangle \times \langle g_{\lambda_2} \rangle \times \cdots \times \langle g_{\lambda_p} \rangle$ .

Notice that  $\{c_\lambda \mid \lambda \text{ is a partition of } n\}$  is a complete set of conjugacy class representatives in  $W$ .

With  $\lambda$  as above, for  $1 \leq i \leq p$ , define  $\phi_{\lambda_i}$  to be the character of  $\langle g_{\lambda_i} \rangle$  with  $\phi_{\lambda_i}(g_{\lambda_i}^{-1}) = e^{2\pi i/\lambda_i}$ . Then  $\phi_{\lambda_i}$  is the analog of the character  $\phi$  in §4 for the factor  $S_{\lambda_i}$  of  $W_\lambda$ . Next, define the

character  $\phi_\lambda$  of  $Z_\lambda \cong \langle g_{\lambda_1} \rangle \times \langle g_{\lambda_2} \rangle \times \cdots \times \langle g_{\lambda_p} \rangle$  to be

$$\phi_\lambda = \phi_{\lambda_1} \otimes \cdots \otimes \phi_{\lambda_p}.$$

Note that this notation is not consistent with that of Lehrer and Solomon; our character  $\phi_\lambda$  corresponds to the character  $\phi_\lambda \epsilon$  in [14]. Applying the special case  $\lambda = (n)$  considered in §4 to each factor  $S_{\lambda_i}$  of  $W_\lambda$ , for  $1 \leq i \leq p$  define

$$f_{\lambda_i}^+ = \frac{1}{\lambda_i} \sum_{k=0}^{\lambda_i-1} \phi_{\lambda_i}(g_{\lambda_i}^k) g_{\lambda_i}^{-k} \quad \text{and} \quad f_{\lambda_i}^- = \frac{1}{\lambda_i} \sum_{k=0}^{\lambda_i-1} \epsilon(g_{\lambda_i}^k) \phi(g_{\lambda_i}^k) g_{\lambda_i}^{-k}.$$

Finally, define idempotents  $f_\lambda^+$  and  $f_\lambda^-$  in  $\mathbb{C}Z_\lambda$  by

$$f_\lambda^+ = f_{\lambda_1}^+ f_{\lambda_2}^+ \cdots f_{\lambda_p}^+ \quad \text{and} \quad f_\lambda^- = f_{\lambda_1}^- f_{\lambda_2}^- \cdots f_{\lambda_p}^-.$$

Obviously the lines  $\mathbb{C}f_\lambda^+$  and  $\mathbb{C}f_\lambda^-$  in  $\mathbb{C}W$  are stable under left and right multiplication by  $Z_\lambda$  and afford the characters  $\phi_\lambda$  and  $\epsilon\phi_\lambda$  of  $Z_\lambda$  respectively.

Now consider the canonical complement  $N_{X_\lambda}$  of  $W_\lambda$  in  $N_W(W_\lambda)$ . Set  $N_\lambda = N_{X_\lambda}$ . If  $\lambda$  has  $m_j$  parts equal  $j$ , then  $N_\lambda$  is isomorphic to the product of symmetric groups  $\prod_j S_{m_j}$  (see [10] or [14]). In particular,  $N_\lambda$  has one Coxeter generator, say  $r_i$ , for each  $i$  such that  $\lambda_i = \lambda_{i+1}$ . The generator  $r_i$  acts on the set  $\{v_1, v_2, \dots, v_n\}$  by interchanging  $v_{\tau_{i-1}+j}$  and  $v_{\tau_i+j}$  for  $1 \leq j \leq \lambda_i$ , and fixing  $v_k$  for  $k \leq \tau_{i-1}$  and  $k > \tau_{i+1}$ .

It is well-known and easy to check ([14], [12]) that  $N_\lambda \subseteq Z_W(c_\lambda)$ , and so  $Z_W(c_\lambda) \cong Z_\lambda \rtimes N_\lambda$ .

**Lemma 5.1.** *The subgroup  $N_\lambda$  of  $Z_W(c_\lambda)$  stabilizes the characters  $\phi_\lambda$  and  $\epsilon\phi_\lambda$  of  $Z_\lambda$ , and centralizes the idempotents  $f_\lambda^+$  and  $f_\lambda^-$ . In particular,  $\phi_\lambda$  extends to a character, also denoted by  $\phi_\lambda$ , of  $Z_W(c_\lambda)$ , with  $\phi_\lambda(nz) = \phi_\lambda(z)$  for  $n$  in  $N_\lambda$  and  $z$  in  $Z_\lambda$ .*

*Proof.* Suppose that  $i$  is such that  $\lambda_i = \lambda_{i+1}$  and consider the generator  $r_i$  of  $N_\lambda$ . Then  $r_i$  is an involution and it follows from the description of the action of  $r_i$  on the basis  $\{v_1, \dots, v_n\}$  of  $V$  that

$$r_i g_{\lambda_j} r_i^{-1} = r_i g_{\lambda_j} r_i = \begin{cases} g_{\lambda_{i+1}} & j = i \\ g_{\lambda_i} & j = i + 1 \\ g_{\lambda_j} & j \neq i, i + 1. \end{cases}$$

Since  $\phi_\lambda(g_{\lambda_i}) = \phi_\lambda(g_{\lambda_{i+1}})$ , it follows that  $r_i$  stabilizes  $\phi_\lambda$  and  $\epsilon\phi_\lambda$ .

The group  $N_\lambda$  is generated by  $\{r_i \mid \lambda_i = \lambda_{i+1},\}$  and so  $N_\lambda$  stabilizes the characters  $\phi_\lambda$  and  $\epsilon\phi_\lambda$  of  $Z_\lambda$ . Moreover,  $N_\lambda$  acts on  $\{g_{\lambda_1}, \dots, g_{\lambda_p}\}$  by conjugation as a group of permutations. Thus, it follows from the definition of  $f_{\lambda_i}^+$  and  $f_{\lambda_i}^-$  that conjugation by  $N_\lambda$  permutes  $\{f_{\lambda_1}^+, \dots, f_{\lambda_p}^+\}$  and  $\{f_{\lambda_1}^-, \dots, f_{\lambda_p}^-\}$ . Since the  $f_{\lambda_i}^+$ 's pairwise commute and the  $f_{\lambda_i}^-$ 's pairwise commute, we see that  $N_\lambda$  centralizes  $f_{\lambda_1}^+ \cdots f_{\lambda_p}^+ = f_\lambda^+$  and  $f_{\lambda_1}^- \cdots f_{\lambda_p}^- = f_\lambda^-$ .  $\square$

Set  $\alpha_\lambda = \alpha_{X_\lambda}$ . Then  $\alpha_\lambda$  is a character of  $N_W(W_\lambda)$  and  $\alpha_\lambda(r_i) = -1$ . Note that this notation is not consistent with that of Lehrer and Solomon; our character  $\alpha_\lambda$  corresponds to the character  $\alpha_\lambda \epsilon$  in [14] as  $\epsilon(r_i) = (-1)^{\lambda_i}$ .

**Theorem 5.2.** *Suppose that  $\lambda$  is a partition of  $n$ . Then the  $N_W(W_\lambda)$ -modules  $e_{I_\lambda} \mathbb{C}W_\lambda$  and  $A_{X_\lambda}$ , and the character  $\phi_\lambda$  of  $Z_W(C_\lambda)$ , are related by*

- (a) the character of the right  $N_W(W_\lambda)$ -module  $e_{I_\lambda} \mathbb{C}W_\lambda$  is  $\text{Ind}_{Z_W(c_\lambda)}^{N_W(W_\lambda)}(\phi_\lambda)$  and
- (b) the character of the left  $N_W(W_\lambda)$ -module  $A_{X_\lambda}$  is  $\epsilon \alpha_\lambda \text{Ind}_{Z_W(c_\lambda)}^{N_W(W_\lambda)}(\phi_\lambda)$ .

*Proof.* Statement (b) has been proved by Lehrer and Solomon [14, Theorem 4.4]. Their argument may be rephrased as follows. Extending the definition of the element  $a_n$  in  $A_n$  when  $\lambda = (n)$ , Lehrer and Solomon define an element  $a_\lambda$  in  $A_{X_\lambda}$  on which  $f_\lambda^-$  acts invertibly. Then:

- (i)  $Z_W(c_\lambda)$  acts on the line  $\mathbb{C}f_\lambda^- a_\lambda$  in  $A_{X_\lambda}$  via the character  $\epsilon_\lambda \alpha_\lambda \phi_\lambda$ .
- (ii)  $A_{X_\lambda} = \mathbb{C}N_W(W_\lambda) f_\lambda^- a_\lambda$ .
- (iii) The multiplication map  $\mathbb{C}N_W(W_\lambda) \otimes_{\mathbb{C}Z_W(c_\lambda)} \mathbb{C}f_\lambda^- a_\lambda \rightarrow A_{X_\lambda}$  is an isomorphism.

Therefore,  $A_{X_\lambda} \cong \text{Ind}_{Z_W(c_\lambda)}^{N_W(W_\lambda)}(\mathbb{C}f_\lambda^- a_\lambda)$  and the character of  $A_{X_\lambda}$  is indeed  $\text{Ind}_{Z_W(c_\lambda)}^{N_W(W_\lambda)}(\epsilon_\lambda \alpha_\lambda \phi_\lambda)$ . Our proof of (a) follows a similar line of reasoning, with  $e_{I_\lambda}$  in place of  $a_\lambda$ .

For the rest of this proof we fix a partition  $\lambda = (\lambda_1, \dots, \lambda_p)$  of  $n$ . To simplify the notation, set  $I = I_\lambda$ . It suffices to show that the line  $\mathbb{C}e_I f_\lambda^+$  in the right  $N_W(W_\lambda)$ -module  $e_I \mathbb{C}W_\lambda$  satisfies properties analogous to (i), (ii), and (iii) above.

- (i')  $Z_W(c_\lambda)$  acts on the line  $\mathbb{C}e_I f_\lambda^+$  via the character  $\phi_\lambda$ : We have  $e_I = x_I e_I^I = x_I e_{\lambda_1}^{\lambda_1} \cdots e_{\lambda_p}^{\lambda_p}$ , where  $e_{\lambda_i}^{\lambda_i}$  in  $S_{\lambda_i}$  is defined using the partition  $(\lambda_i)$  of  $\lambda_i$ . By Lemma 4.5, the idempotent  $f_{\lambda_i}^+$  acts invertibly by right multiplication on  $e_{\lambda_i}^{\lambda_i}$  for  $1 \leq i \leq p$ . Since

$$e_I f_\lambda^+ = x_I (e_{\lambda_1}^{\lambda_1} \cdots e_{\lambda_p}^{\lambda_p}) (f_{\lambda_1}^+ \cdots f_{\lambda_p}^+) = x_I (e_{\lambda_1}^{\lambda_1} f_{\lambda_1}^+) \cdots (e_{\lambda_p}^{\lambda_p} f_{\lambda_p}^+),$$

it follows that  $f_\lambda^+$  acts invertibly on  $e_I$ . In particular,  $e_I f_\lambda^+ \neq 0$ . Moreover, it is clear that  $Z_\lambda$  acts on  $\mathbb{C}e_I f_\lambda^+$  via the character  $\phi_\lambda$ . We have seen in Lemma 3.1 that  $e_I n = e_I$  for  $n$  in  $N_\lambda$ . Thus, to show that  $Z_W(c_\lambda)$  acts on  $\mathbb{C}e_I f_\lambda^+$  via the character  $\phi_\lambda$ , it is enough to show that  $N_\lambda$  centralizes  $f_\lambda^+$  in  $\mathbb{C}W$ , but this was shown in Lemma 5.1.

- (ii')  $e_I \mathbb{C}W_\lambda = e_I f_\lambda^+ \mathbb{C}N_W(W_\lambda)$ : Because  $e_I N_\lambda = e_I$  and  $f_\lambda^+$  acts invertibly on  $e_I$ , we have

$$e_I \mathbb{C}W_\lambda = e_I \mathbb{C}N_W(W_\lambda) = e_I f_\lambda^+ \mathbb{C}N_W(W_\lambda).$$

- (iii') The multiplication map  $\mathbb{C}e_I f_\lambda^+ \otimes_{\mathbb{C}Z_W(c_\lambda)} \mathbb{C}N_W(W_\lambda) \rightarrow e_I \mathbb{C}W_\lambda$  is an isomorphism: It follows from (ii') that the mapping is surjective. Moreover,

$$\begin{aligned} \dim e_I f_\lambda^+ \mathbb{C}N_W(W_\lambda) &= \dim e_I \mathbb{C}W_\lambda \\ &= \dim e_I^I \mathbb{C}W_\lambda \\ &= |W_\lambda : Z_\lambda| \\ &= |N_W(W_\lambda) : Z_W(c_\lambda)| \\ &= \dim \mathbb{C}e_I f_\lambda^+ \otimes_{\mathbb{C}Z_W(c_\lambda)} \mathbb{C}N_W(W_\lambda) \end{aligned}$$

and so the mapping is an isomorphism.



Therefore,  $e_I \mathbb{C}W_\lambda \cong \text{Ind}_{Z_W(c_\lambda)}^{N_W(W_\lambda)}(\mathbb{C}e_I f_\lambda^+)$  and the character of  $e_I \mathbb{C}W_\lambda$  is  $\text{Ind}_{Z_W(c_\lambda)}^{N_W(W_\lambda)}(\phi_\lambda)$  as claimed.  $\square$

The proof of Conjecture 2.1 for symmetric groups now follows from Proposition 3.6, Theorem 5.2, and transitivity of induction.

**Theorem 5.3.** *For each partition  $\lambda$  of  $n$  there is a linear character  $\phi_\lambda$  of  $Z_W(c_\lambda)$  such that*

- (a) *the character of  $E_\lambda$  is  $\text{Ind}_{Z_W(c_\lambda)}^W(\phi_\lambda)$  and*
- (b) *the character of  $A_\lambda$  is  $\text{Ind}_{Z_W(c_\lambda)}^W(\epsilon_\lambda \alpha_\lambda \phi_\lambda)$ , where  $\epsilon_\lambda$  denotes the restriction of  $\epsilon$  to  $Z_W(c_\lambda)$ .*

*In particular,*

$$H^p(M) \cong \bigoplus_{\lambda \vdash n, \text{rk}(c_\lambda)=p} \text{Ind}_{Z_W(c_\lambda)}^W(\epsilon_\lambda \alpha_\lambda \phi_\lambda)$$

*for  $0 \leq p \leq n - 1$ , and*

$$\mathbb{C}W \cong \bigoplus_{\lambda \vdash n} \text{Ind}_{Z_W(c_\lambda)}^W(\phi_\lambda) \quad \text{and} \quad A \cong \bigoplus_{\lambda \vdash n} \text{Ind}_{Z_W(c_\lambda)}^W(\epsilon_\lambda \alpha_\lambda \phi_\lambda).$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, DENTON TX, USA 76203

*E-mail address:* [douglass@unt.edu](mailto:douglass@unt.edu)

*URL:* <http://hilbert.math.unt.edu>

SCHOOL OF MATHEMATICS, STATISTICS AND APPLIED MATHEMATICS, NATIONAL UNIVERSITY OF IRELAND, GALWAY, UNIVERSITY ROAD, GALWAY, IRELAND

*E-mail address:* [goetz.pfeiffer@nuigalway.ie](mailto:goetz.pfeiffer@nuigalway.ie)

*URL:* <http://schmidt.nuigalway.ie/~goetz>

FAKULTÄT FÜR MATHEMATIK, RUHR-UNIVERSITÄT BOCHUM, D-44780 BOCHUM, GERMANY

*E-mail address:* [gerhard.roehrle@rub.de](mailto:gerhard.roehrle@rub.de)

*URL:* <http://www.ruhr-uni-bochum.de/ffm/Lehrstuehle/Lehrstuhl-VI>