

# ON REFLECTION SUBGROUPS OF FINITE COXETER GROUPS

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ABSTRACT. Let  $W$  be a finite Coxeter group. We classify the reflection subgroups of  $W$  up to conjugacy and give necessary and sufficient conditions for the map that assigns to a reflection subgroup  $R$  of  $W$  the conjugacy class of its Coxeter elements to be injective, up to conjugacy.

## 1. INTRODUCTION

Throughout, let  $(W, S)$  be a finite Coxeter system with distinguished set of generators  $S$  and let  $E$  be the real reflection representation of  $W$ . Define  $T = \{ws w^{-1} \mid w \in W, s \in S\}$  to be the set of elements of  $W$  that act on  $E$  as reflections. By a *reflection subgroup* of  $W$  we mean a subgroup of  $W$  generated by a subset of  $T$ . Reflection subgroups of  $W$  play an important role in the theory of Coxeter groups; for instance, by a fundamental theorem due to Steinberg, [13, Thm. 1.5], the stabilizer of any subspace of  $E$  is a reflection subgroup of  $W$ .

Our first aim in this note is to give a complete classification of all reflection subgroups of  $W$  up to conjugacy. In case  $W$  is a Weyl group, Carter [5, p. 8] has already outlined a procedure which leads to the classification based on the algorithm of Borel–De Siebenthal [3]. Here, we recast slightly Carter’s construction and give the classification for non-crystallographic Coxeter groups as well.

Every reflection subgroup of  $W$  is a maximal rank reflection subgroup of some parabolic subgroup of  $W$ . Thus, classifying conjugacy classes of reflection subgroups may be done recursively and reduces to first classifying conjugacy classes of parabolic subgroups and then classifying maximal rank subgroups of irreducible Coxeter groups. Conjugacy classes of parabolic subgroups of an irreducible finite Coxeter group are described in Chapter 2 and Appendix A of [8]. Classifying maximal rank reflection subgroups of  $W$  amounts to listing, up to the action of  $W$ , all subsets  $Y$  of  $T$  whose fixed point set in  $E$  is trivial and which are closed in the sense that  $\langle Y \rangle \cap T = Y$ . In case  $W$  is a Weyl group, the algorithm of Borel–De Siebenthal [3] is computationally much more efficient than classifying subsets of  $T$  with the two required properties.

We have implemented the classification algorithms in the computer algebra system GAP [11] with the aid of the package CHEVIE [9]. Thus, it is feasible to actually compute the classification explicitly for  $W$  of a fixed rank. Indeed, we present the classification in cases  $W$  is

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a Weyl group of exceptional type, or a non-crystallographic Coxeter group of type  $H_3$  and  $H_4$ , in the form of explicit lists.

Our second aim in this note is to study the map which assigns to a given conjugacy class of reflection subgroups the conjugacy class of its Coxeter elements. It is well known [10, Lem. 3.5] that if  $R$  and  $R'$  are parabolic subgroups containing Coxeter elements  $c$  and  $c'$  respectively, then  $R$  and  $R'$  are conjugate subgroups if and only if  $c$  and  $c'$  are conjugate in  $W$ . Thus, conjugacy classes of parabolic subgroups are parametrized by a distinguished set of conjugacy classes of elements in  $W$ . For general reflection subgroups this need not be the case. However, the following somewhat surprising result shows that when  $T$  is a single conjugacy class, conjugacy classes of reflection subgroups are still parametrized by the conjugacy classes of their Coxeter elements in all but one case.

**Theorem 1.1.** *Suppose that  $T$  is a single conjugacy class. Let  $R$  and  $R'$  be parabolic subgroups containing Coxeter elements  $c$  and  $c'$ , respectively. Then  $R$  and  $R'$  are conjugate if and only if  $c$  and  $c'$  are conjugate in  $W$ ; unless  $W$  is of type  $E_8$  and  $R$  and  $R'$  are of types  $A_1A_7$  and  $A_3D_5$ , respectively. In this case,  $c$  and  $c'$  are conjugate, while  $R$  and  $R'$  are not.*

This theorem is an immediate consequence of the classification of the reflection subgroups of  $W$  and our computation of the map  $\gamma$ , which is defined as follows. Denote by  $\mathcal{R}$  the set of conjugacy classes of reflection subgroups of  $W$  and by  $\mathcal{C}$  the set of conjugacy classes of elements of  $W$ . Then, denote by

$$\gamma: \mathcal{R} \rightarrow \mathcal{C}$$

the map defined by  $\gamma([R]) = [c]$ , which associates to the conjugacy class  $[R]$  of a reflection subgroup  $R$  of  $W$  the conjugacy class  $[c]$  in  $W$  of a Coxeter element  $c$  in  $R$ . In §2 we state necessary and sufficient conditions for the map  $\gamma$  to be an injection. Also, the image of  $\gamma$  is computed explicitly for each type of irreducible Coxeter group in §3 - §5 and Tables 3 - 9. The classes in the image of  $\gamma$  are also known in the literature as *semi-Coxeter classes*, e.g., see [6].

The rest of this note is organized as follows. In §2 we recall some definitions, give some preliminary results, and state precisely when the map  $\gamma$  is injective. The classification of conjugacy classes of reflection subgroups and the explicit computation of the map  $\gamma$  is given for classical Weyl groups in §3; for exceptional Weyl groups in §4 and Tables 3 - 7; and for non-crystallographic Coxeter groups in §5 and Tables 8 and 9.

## 2. PRELIMINARIES

For general information on Coxeter groups, groups generated by reflections and root systems, we refer the reader to Bourbaki [4].

For the rest of this note we fix a  $W$ -invariant, positive, definite bilinear form on  $E$ .

Notice first that if  $R = \langle Y \rangle$  is a reflection subgroup of  $W$ , then  $R$  is a Coxeter group in its own right. Moreover, the orthogonal complement of the space of fixed points of  $R$  in  $E$  is an  $R$ -stable subspace that affords the reflection representation of  $R$ .

Recall that a *parabolic subgroup* of  $W$  is a subgroup of the form

$$W_V = \{ w \in W \mid w(v) = v \forall v \in V \},$$

where  $V$  is a subspace of  $E$ . By Steinberg's Theorem [13, Thm. 1.5], parabolic subgroups are generated by the reflections they contain and so are reflection subgroups.

For a subset  $X$  of  $W$  let

$$\text{Fix}(X) = \{ v \in E \mid x(v) = v \forall x \in X \}$$

denote the set of fixed points of  $X$  in  $E$ . Following Solomon [12] and Bergeron et al. [2], we define the *parabolic closure* of  $X$  to be the parabolic subgroup  $A(X) = W_{\text{Fix}(X)}$  of  $W$ . Obviously  $X \subseteq A(X)$  and it follows from Steinberg's Theorem that  $A(A(X)) = A(X)$ . When  $X = \{w\}$  we write simply  $\text{Fix}(w)$  and  $A(w)$  instead of  $\text{Fix}(\{w\})$  and  $A(\{w\})$ , respectively. For a discussion of parabolic closures of finitely generated subgroups of arbitrary Coxeter systems, see the recent paper by Dyer [7].

The *rank* of a Coxeter group is the cardinality of a Coxeter generating set, or equivalently, the dimension of its reflection representation. It follows from the next lemma that every reflection subgroup is a maximal rank reflection subgroup of its parabolic closure.

**Lemma 2.1.** *Let  $R$  be a reflection subgroup of  $W$ . Then  $R$  and its parabolic closure  $A(R)$  have the same rank as Coxeter groups.*

*Proof.* The rank of  $R$  is the codimension of its fixed point space  $\text{Fix}(R)$ . The rank of  $A(R)$ , as stabilizer of  $\text{Fix}(R)$ , is not larger than the rank of  $R$ , and, since  $R \subseteq A(R)$ , not smaller than the rank of  $R$  either.  $\square$

As noted in the Introduction, the classification of conjugacy classes of reflection subgroups of  $W$  reduces to (1) classifying conjugacy classes of parabolic subgroups of  $W$  and (2) classifying maximal rank reflection subgroups of irreducible Coxeter groups.

The conjugacy classes of parabolic subgroups of an irreducible finite Coxeter group are described in Chapter 2 and Appendix A of [8] (see also [1, Prop. 6.3]). In most cases, two parabolic subgroups are conjugate if and only if they have the same type. However, in type  $D_{2m}$  there are two conjugacy classes of parabolic subgroups of type  $A_{k_1} \times A_{k_2} \times \cdots \times A_{k_r}$  with all  $k_i$  odd so that  $2m = \sum(k_i + 1)$  and in type  $E_7$  there are two classes of parabolic subgroups of type  $A_1^3$ ,  $A_1A_3$ , as well as  $A_5$ .

For a given  $W$ , classifying the maximal rank reflection subgroups of  $W$  up to conjugacy amounts to listing, up to conjugacy in  $W$ , all subsets  $Y$  of  $T$  such that

$$\langle Y \rangle \cap T = Y \quad \text{and} \quad \text{Fix}(Y) = \{0\}.$$

For a Coxeter group of small rank (including the non-crystallographic types  $H_3$  and  $H_4$ ) these sets can be systematically enumerated. As described below, for crystallographic Coxeter groups, that is, Weyl groups, using the algorithm of Borel–De Siebenthal [3] is computationally more efficient than classifying subsets of  $T$ .

By a *root system in  $E$*  we mean a reduced root system in the sense of Bourbaki [4, Ch. VI]. Suppose  $\Phi$  is a root system in  $E$ . The Weyl group of  $\Phi$ ,  $W(\Phi)$ , is the group of linear transformations of  $E$  generated by the reflections through the hyperplanes orthogonal to the roots in  $\Phi$ . The *dual* of  $\Phi$  is the root system  $\tilde{\Phi} = \{ \frac{1}{|\alpha|^2} \alpha \mid \alpha \in \Phi \}$ . Note that  $W(\Phi) = W(\tilde{\Phi})$ . By a *Weyl group* or a *crystallographic Coxeter group* we mean the Weyl group of a root system in  $E$ .

Suppose that  $W = W(\Phi) = W(\tilde{\Phi})$  is a Weyl group. We may extract a classification of the maximal rank reflection subgroups of  $W$  from the arguments in [5]. Each maximal rank reflection subgroup of  $W$  is again a Weyl group and thus is the Weyl group of a maximal rank subsystem of  $\Phi$  or  $\tilde{\Phi}$ . By work of Dynkin, two maximal rank subsystems are isomorphic if and only if they are equivalent under the action of  $W$ ; see [5, Prop. 32] or [4, Ch. VI, §4, ex. 4]. By the classification of root systems, two root systems are isomorphic if and only if they have the same Dynkin diagram. We have already observed that a root system and its dual have the same Weyl group. Thus, the conjugacy classes of maximal rank reflection subgroups of  $W$  are in one-one correspondence with the set of Coxeter graphs arising from Dynkin diagrams of maximal rank subsystems of  $\Phi$  and  $\tilde{\Phi}$ .

The Borel–De Siebenthal algorithm produces all maximal rank subsystems of  $\Phi$  and  $\tilde{\Phi}$  as follows (see [5, p. 8]).

- (1) Add a node to the Dynkin diagram of  $\Phi$  corresponding to the negative of the highest root of  $\Phi$ . Take the extended Dynkin diagram and remove one node in all possible ways.
- (2) Add a node to the Dynkin diagram of  $\tilde{\Phi}$  corresponding to the negative of the highest root of  $\tilde{\Phi}$ . Take the extended Dynkin diagram and remove one node in all possible ways.
- (3) Repeat steps (1) and (2) with each of the resulting Dynkin diagrams until no new diagrams appear.

We now turn to the map  $\gamma$  which assigns to a given conjugacy class of reflection subgroups of  $W$  the conjugacy class of its Coxeter elements.

Recall [4, Ch. V, §6, no. 1] that a *Coxeter element* in  $W$  is the product of the elements of some Coxeter generating set of  $W$  taken in some order. All Coxeter elements of  $W$  are conjugate in  $W$ .

Suppose that  $R$  is a reflection subgroup of  $W$ . Then  $R$  is a Coxeter group and so we may consider Coxeter elements in  $R$ . If  $c$  is a Coxeter element in  $R$  and  $w$  is in  $W$ , it is easy to see that  $c^w$  is a Coxeter element in  $R^w$ . Thus, conjugate reflection subgroups of  $W$  have conjugate Coxeter elements and the map  $\gamma$  is well-defined.

The proof of Theorem 1.1 follows immediately from the classification of reflection subgroups and the explicit computation of the map  $\gamma$  in Theorems 3.1 and 3.3 and Tables 3 - 5. It would be interesting to have a conceptual explanation of why the single exception occurs in

**Theorem 1.1.** More generally, from Theorems 3.1, 3.2, and 3.3 along with Tables 3 - 9, we derive necessary and sufficient conditions for the map  $\gamma$  to be injective.

**Theorem 2.2.** *Suppose that  $W$  is irreducible and not of type  $E_8$ . Then the map  $\gamma: \mathcal{R} \rightarrow \mathcal{C}$  is injective if and only if  $T$  is a single conjugacy class in  $W$ .*

*If  $W$  is of type  $E_8$ , then  $\gamma$  is not injective: the conjugacy classes of reflection subgroups of types  $A_1A_7$  and  $A_3D_5$  both map to the same conjugacy class of elements of  $W$ .*

Notice that when the map  $\gamma$  is surjective, every conjugacy class of  $W$  contains a representative that is a Coxeter element in some reflection subgroup of  $W$ . It follows from the computations in §3 - §5 that, if  $W$  is irreducible, then  $\gamma$  is surjective when  $W$  has one of the following types:  $A_n$ ,  $B_n$ , or  $G_2$ .

It is easy to see that  $A(X^w) = A(X)^w$  when  $X \subseteq W$  and  $w \in W$ . In particular, conjugate reflection subgroups have conjugate parabolic closures. Similarly, conjugate elements in  $W$  have conjugate parabolic closures. In particular, if  $c \in R$  and  $c' \in R'$  are Coxeter elements in reflection subgroups  $R$  and  $R'$ , and  $c$  and  $c'$  are conjugate in  $W$ , then  $A(c)$  and  $A(c')$  are conjugate in  $W$ .

**Lemma 2.3.** *Suppose  $R$  is a reflection subgroup of  $W$  and  $x \in R$  is not contained in any proper parabolic subgroup of  $R$ . Then  $A(x) = A(R)$ . Consequently, if  $V$  is a subspace of  $E$  and  $c$  is a Coxeter element of  $R$  that is conjugate to an element of  $W_V$ , then  $R$  is conjugate to a subgroup of  $W_V$ .*

*Proof.* It is shown in [5, §2] that  $\text{Fix}(x) = \text{Fix}(R)$ . It follows immediately that  $A(x) = A(R)$ .

For the second statement, let  $w \in W$  be such that  $c^w$  is in  $W_V$ . Then  $A(c^w) \subseteq W_V$  and so  $R^w \subseteq A(R)^w = A(c)^w = A(c^w) \subseteq W_V$ .  $\square$

Now suppose that  $R$  and  $R'$  are reflection subgroups of  $W$  containing Coxeter elements  $c$  and  $c'$  respectively. Then, if  $c$  and  $c'$  are conjugate in  $W$ ,  $A(R)$  and  $A(R')$  are conjugate. In other words, even if  $\gamma$  is not injective, reflection subgroups with non-conjugate parabolic closures must have non-conjugate Coxeter elements. This observation shows that conjugacy classes containing Coxeter elements of reflection subgroups are separated by the parabolic closures of reflection subgroups that contain them.

Note that Lemma 2.3 generalizes [12, Lem. 7] which is the special case of Lemma 2.3 when  $R$  is a parabolic subgroup of  $W$ . In the same way, Theorem 1.1 generalizes the forward implication of [10, Lem. 3.5].

### 3. THE CLASSICAL WEYL GROUPS

A *partition*  $\lambda = (\lambda_1, \dots, \lambda_k)$  is a non-increasing finite sequence of positive integers  $\lambda_1 \geq \dots \geq \lambda_k > 0$ . The integers  $\lambda_i$  are called the *parts* of the partition  $\lambda$ . If  $\sum_{i=1}^k \lambda_i = n$ , then  $\lambda$  is a partition of  $n$  and we write  $\lambda \vdash n$ . The unique partition of  $n = 0$  is the *empty partition*, denoted by  $\emptyset$ . We denote by  $\ell(\lambda) = k$  the length of the partition  $\lambda = (\lambda_1, \dots, \lambda_k)$ ,

e.g.,  $\ell(\emptyset) = 0$ . A partition of  $n$  is *even*, if all its parts are even, i.e., if it has the form  $\lambda = (2\mu_1, \dots, 2\mu_k)$  for some partition  $\mu$  of  $n/2$ . The *join*  $\lambda^1 \cup \lambda^2$  of two partitions  $\lambda^1 \vdash n_1$  and  $\lambda^2 \vdash n_2$  is the partition of  $n_1 + n_2$  consisting of the parts of both  $\lambda^1$  and  $\lambda^2$ , suitably arranged. The *sum* of a partition  $\lambda = (\lambda_1, \dots, \lambda_k)$  and an integer  $m$  is the partition  $\lambda + m = (\lambda_1 + m, \dots, \lambda_k + m)$ . We write  $\lambda > m$  if  $\lambda_i > m$  for all parts  $\lambda_i$  of  $\lambda$ . Note that, vacuously,  $\emptyset > m$  for all  $m$ .

The symmetric group  $\mathfrak{S}_n$  on  $n$  points is a Coxeter group of type  $A_{n-1}$  with Coxeter generators  $s_i = (i, i+1)$ ,  $i = 1, \dots, n-1$ . The *cycle type* of  $w \in \mathfrak{S}_n$  is the partition  $\lambda$  of  $n$  which has the lengths of the cycles of  $w$  on  $\{1, \dots, n\}$  as its parts (here a fixed point contributes a cycle of length 1). Of course, two permutations in  $\mathfrak{S}_n$  are conjugate if and only if they have the same cycle type. The next theorem is well-known, and can easily be deduced from Bourbaki [4, Ch. VI, §4, ex. 4].

**Theorem 3.1.** *Let  $W$  be a Coxeter group of type  $A_n$ . Then every reflection subgroup of  $W$  is a parabolic subgroup. Moreover, the map  $\gamma$  from conjugacy classes of reflection subgroups to conjugacy classes of  $W$  is a bijection. Both sets are in one-to-one correspondence with the set of all partitions of  $n$ .*

An  $r$ -*partition* is a sequence  $\lambda = (\lambda^1, \dots, \lambda^r)$  of  $r$  partitions  $\lambda^1, \dots, \lambda^r$ . We say that  $\lambda$  is an  $r$ -partition of the integer  $n$ , and write  $\lambda \vdash^r n$ , if  $\lambda^1 \cup \dots \cup \lambda^r \vdash n$ . We call  $\lambda$  a *double partition* if  $r = 2$ , and a *triple partition* if  $r = 3$ .

The Coxeter group  $W(B_n)$  acts faithfully as a group of signed permutations on the set of long roots  $\{\pm e_i : i = 1, \dots, n\}$ , permuting the lines  $\langle e_i \rangle$ ,  $i = 1, \dots, n$ . A cycle of  $w$  in  $W(B_n)$  is either *positive* or *negative*, depending on whether the number of positive roots  $e_i$  with  $\langle e_i \rangle$  in the cycle that are mapped to a negative root is even or odd. The *cycle type* of  $w$  in  $W(B_n)$  is a double partition  $\lambda = (\lambda^1, \lambda^2)$  of  $n$ , where  $\lambda^1$  records the lengths of the positive cycles of  $w$  and  $\lambda^2$  records the lengths of the negative cycles. Again, two elements of  $W(B_n)$  are conjugate if and only if they have the same cycle type and, in this way, double partitions of  $n$  naturally parametrize the conjugacy classes of  $W(B_n)$ .

According to [8, Prop. 2.3.10], the parabolic subgroups of  $W(B_n)$  are of the form  $W(B_{n-m}) \times \prod_i W(A_{\lambda_i-1})$ , one conjugacy class for each partition  $\lambda \vdash m$ ,  $0 \leq m \leq n$ . By Borel–De Siebenthal, the maximal rank reflection subgroups of  $W(B_n)$  are of type  $\prod_i W(B_{\lambda_i^1}) \times \prod_i W(D_{\lambda_i^2})$ , one class for each double partition  $\lambda \vdash^2 n$  with  $\lambda^2 > 1$  (or  $\lambda^2 = \emptyset$ ). It follows that the reflection subgroups of  $W(B_n)$  are direct products of Coxeter groups of types  $A$ ,  $B$  and  $D$ , and their classes are naturally labeled by triple partitions of  $n$ .

**Theorem 3.2.** *Let  $W$  be a Coxeter group of type  $B_n$ . Then the conjugacy classes of reflection subgroups of  $W$  are represented by*

$$\{W_\lambda : \lambda \vdash^3 n, \lambda^3 > 1\},$$

where  $W_\lambda = \prod_i W(A_{\lambda_i^1-1}) \times \prod_i W(B_{\lambda_i^2}) \times \prod_i W(D_{\lambda_i^3})$ . The parabolic closure of  $W_\lambda$  has type  $W(B_{n-m}) \times \prod_i W(A_{\lambda_i^1-1})$ , where  $\lambda^1 \vdash m$ . The Coxeter elements of  $W_\lambda$  have cycle type  $(\lambda^1, \lambda^2 \cup (\lambda^3 - 1) \cup 1^{\ell(\lambda^3)})$ . In particular, the map  $\gamma : \mathcal{R} \rightarrow \mathcal{C}$  is surjective.



We illustrate the classification in type  $B_n$  in Table 1 below, where we list all conjugacy classes of reflection subgroups of  $W(B_5)$  according to Theorem 3.2. Clearly, it follows from the data in Table 1 that  $\gamma$  is not injective in this case.

The Coxeter group  $W(D_n)$  is a normal subgroup of index 2 in  $W(B_n)$ , and as such it is a union of  $W(B_n)$ -conjugacy classes of elements. In fact, the class of elements of cycle type  $\lambda = (\lambda^1, \lambda^2)$  is contained in  $W(D_n)$  if and only if  $\ell(\lambda^2)$  is even, and it is a single conjugacy class in  $W(D_n)$ , unless  $\lambda^2 = \emptyset$  and  $\lambda^1$  is even. In the latter case, the  $W(B_n)$ -class splits into two  $W(D_n)$ -classes, labelled  $(\lambda^1, +)$  and  $(\lambda^1, -)$ . In this way, the conjugacy classes of  $W(D_n)$  are parametrized by double partitions of  $n$ .

According to [8, Prop. 2.3.13],  $W(D_n)$  has three distinct kinds of parabolic subgroups: one class of subgroups of type  $W(D_{n-m}) \times \prod_i W(A_{\lambda_i-1})$  for each partition  $\lambda \vdash m$ ,  $0 \leq m \leq n-2$ , two classes of subgroups of type  $\prod_i W(A_{\lambda_i-1})$  for each even partition  $\lambda \vdash n$ , and one class of subgroups of type  $\prod_i W(A_{\lambda_i-1})$  for each non-even partition  $\lambda \vdash n$ . Reflection subgroups of  $W(D_n)$  are direct products of Coxeter groups of types  $A$  and  $D$ , and their classes are naturally labeled by double partitions of  $n$ . By Borel–De Siebenthal, the maximal rank reflection subgroups of  $W(D_n)$  are of type  $\prod_i W(D_{\lambda_i})$ , one class for each partition  $\lambda \vdash n$  with  $\lambda > 1$ . This yields the following classification of the conjugacy classes of reflection subgroups of  $W(D_n)$ , in terms of double partitions of  $n$ .

**Theorem 3.3.** *Let  $W$  be a Coxeter group of type  $D_n$ ,  $n \geq 4$ . Then the conjugacy classes of reflection subgroups of  $W$  are represented by*

$$\{W_\lambda : \lambda \vdash^2 n, \lambda^2 > 1\}$$

*if  $n$  is odd, and by*

$$\{W_\lambda : \lambda \vdash^2 n, \lambda^2 > 1 \text{ and } \lambda^1 \text{ non-even in case } \lambda^2 = \emptyset\} \cup \{W_{2\lambda}^\pm : \lambda \vdash \frac{n}{2}\}$$

*if  $n$  is even, where  $W_\lambda = \prod_i W(A_{\lambda_i^1-1}) \times \prod_i W(D_{\lambda_i^2})$  and  $W_\lambda^\epsilon = \prod_i W(A_{\lambda_i-1})$ . The parabolic closure of  $W_\lambda$  has type  $W(D_{n-m}) \times \prod_i W(A_{\lambda_i^1-1})$ , where  $\lambda^1 \vdash m$ ; the parabolic closure of  $W_\lambda^\epsilon$  is  $W_\lambda^\epsilon$  itself. The Coxeter elements of  $W_\lambda$  have cycle type  $(\lambda^1, (\lambda^2 - 1) \cup 1^{\ell(\lambda^2)})$ . The Coxeter elements of  $W_\lambda^\epsilon$  have cycle type  $(\lambda, \epsilon)$ . In particular, the map  $\gamma: \mathcal{R} \rightarrow \mathcal{C}$  is injective, but not surjective.*

We illustrate the classification in type  $D_n$  from Theorem 3.3 for  $n = 6$  in Table 2 below.

#### 4. THE EXCEPTIONAL WEYL GROUPS

For the exceptional Weyl groups all results are obtained by following the recursive procedure outlined in §2 by applying the Borel–De Siebenthal algorithm for the various factors of each standard parabolic subgroup of  $W$ . The calculations were carried out with the use of GAP [11] and CHEVIE [9]. Here the conjugacy classes of the elements in  $W$  are labeled as in Carter [5].

In Tables 3 - 7 we list all reflection subgroups in case  $W$  is of exceptional type up to conjugacy. In the cases when  $W$  has only a single class of reflections, it is readily checked that  $\gamma$  is injective, as required for Theorem 1.1.

Table 5 contains the results for  $W(E_8)$ . Here the two maximal rank reflection subgroups of types  $A_1A_7$  and  $A_3D_5$  both have Coxeter elements that are conjugate in  $W$ . Hence  $\gamma$  is not injective.

In Tables 6 and 7 we list all conjugacy classes of reflection subgroups of  $W(F_4)$  and  $W(G_2)$ , respectively. In both instances we see that  $\gamma$  is not injective.

## 5. THE NON-CRYSTALLOGRAPHIC CASES

As in the exceptional cases, the non-crystallographic instances were computed using GAP [11] and CHEVIE [9]. The Borel–De Siebenthal algorithm does not apply, but these groups are sufficiently small that all the relevant information can be calculated directly.

In Tables 8 and 9 we list all conjugacy classes of reflection subgroups of  $W(H_3)$  and  $W(H_4)$ , respectively. Here we see that  $\gamma$  is injective in both cases. The labeling of the conjugacy classes is the one used by CHEVIE.

The reflection subgroups of the dihedral group  $W(I_2(m))$  can be described as follows.

**Theorem 5.1.** *Let  $W$  be of type  $I_2(m)$ ,  $m = 5$ , or  $m > 6$ .*

- (1) *If  $m$  is odd then the classes of reflection subgroups of  $W$  are of type  $\emptyset$ ,  $A_1$ , and  $I_2(d)$ , where  $d > 1$  is a divisor of  $m$ . The map  $\gamma$  is injective, but not surjective.*
- (2) *If  $m$  is even then the classes of reflection subgroups of  $W$  are of type  $\emptyset$ ,  $A_1$ ,  $\tilde{A}_1$ ,  $I_2(d)$ , where  $d > 1$  is a divisor of  $m$ , and  $\tilde{I}_2(d)$ , where  $2d > 2$  is a divisor of  $m$ . The map  $\gamma$  is neither injective nor surjective.*

## 6. TABLES

In Tables 1 - 9 we present the classification of the reflection subgroups of  $W$  in various cases. The tables provide the following information. In the first column of each table we list the types of the reflection subgroups  $R$  of  $W$ . In the second column in Tables 1 and 2 we also give the partition characterizing  $R$  according to Theorems 3.2 and 3.3, respectively. The next two columns give the cardinality of  $R$  and the cardinality of the class  $[R]$ , that is,  $|W : N_W(R)|$ , of  $R$ . Finally, in the last column we list the image of  $\gamma$ , i.e. the class  $[c]$  of a Coxeter element  $c$  of  $R$  in  $W$ . For the classical types, conjugacy classes are labeled by cycle type. For the exceptional types, conjugacy classes are labeled as in Carter’s classification [5].

Conjugacy classes of reflection subgroups with distinct parabolic classes are separated by horizontal lines. For a given parabolic subgroup  $P$  of  $W$ , the row for  $P$  is preceded by a horizontal line and followed by the rows for reflection subgroups  $R$  of  $W$  with  $A(R) = P$ .



**Table 1.** Reflection subgroups of  $W(B_5)$ .

Type of $R$	$\lambda$	$ R $	$  R  $	Class	Type of $R$	$\lambda$	$ R $	$  R  $	Class
$\emptyset$	$1^5..$	1	1	$1^5.$	$A_4$	$5..$	120	16	5.
$B_1$	$1^4.1.$	2	5	$1^4.1$	$B_4$	1.4.	384	5	1.4
$A_1$	$21^3..$	2	20	$21^3.$	$B_1B_3$	1.31.	96	20	1.31
$B_1A_1$	$21^2.1.$	4	60	$21^2.1$	$B_2^2$	$1.2^2.$	64	15	$1.2^2$
$A_1^2$	$2^21..$	4	60	$2^21.$	$D_4$	1..4	192	5	1.31
$A_2$	$31^2..$	6	40	$31^2.$	$D_3B_1$	1.1.3	48	20	$1.21^2$
$B_2$	$1^3.2.$	8	10	$1^3.2$	$D_2B_2$	1.2.2	32	30	$1.21^2$
$B_1^2$	$1^3.1^2.$	4	10	$1^3.1^2$	$B_1^2B_2$	$1.21^2.$	32	30	$1.21^2$
$D_2$	$1^3..2$	4	10	$1^3.1^2$	$D_2B_1^2$	$1.1^2.2$	16	30	$1.1^4$
$B_1A_1^2$	$2^2.1.$	8	60	$2^2.1$	$B_1^4$	$1.1^4.$	16	5	$1.1^4$
$B_1A_2$	$31.1.$	12	80	$31.1$	$D_2^2$	$1..2^2$	16	15	$1.1^4$
$A_1A_2$	$32..$	12	80	$32.$	$B_5$	.5.	3840	1	.5
$B_2A_1$	$21.2.$	16	60	$21.2$	$B_1B_4$	.41.	768	5	.41
$B_1^2A_1$	$21.1^2.$	8	60	$21.1^2$	$B_2B_3$	.32.	384	10	.32
$D_2A_1$	$21..2$	8	60	$21.1^2$	$D_5$	..5	1920	1	.41
$A_3$	$41..$	24	40	41.	$D_4B_1$	.1.4	384	5	.31 <sup>2</sup>
$B_3$	$1^2.3.$	48	10	$1^2.3$	$D_3B_2$	.2.3	192	10	.21 <sup>2</sup>
$B_1B_2$	$1^2.21.$	16	30	$1^2.21$	$D_2B_3$	.3.2	192	10	.31 <sup>2</sup>
$D_3$	$1^2..3$	24	10	$1^2.21$	$B_1^2B_3$	.31 <sup>2</sup> .	192	10	.31 <sup>2</sup>
$D_2B_1$	$1^2.1.2$	8	30	$1^2.1^3$	$B_1B_2^2$	.2 <sup>2</sup> 1.	128	15	.21 <sup>2</sup>
$B_1^3$	$1^2.1^3.$	8	10	$1^2.1^3$	$D_3B_1^2$	.1 <sup>2</sup> .3	96	10	.21 <sup>3</sup>
$B_2A_2$	3.2.	48	40	3.2	$D_2B_1B_2$	.21.2	64	30	.21 <sup>3</sup>
$B_1^2A_2$	$3.1^2.$	24	40	$3.1^2$	$B_1^3B_2$	.21 <sup>3</sup> .	64	10	.21 <sup>3</sup>
$D_2A_2$	3..2	24	40	$3.1^2$	$D_2B_1^3$	.1 <sup>3</sup> .2	32	10	.1 <sup>5</sup>
$B_1A_3$	4.1.	48	40	4.1	$D_2^2B_1$	.1.2 <sup>2</sup>	32	15	.1 <sup>5</sup>
$B_3A_1$	2.3.	96	20	2.3	$D_2D_3$	..23	96	10	.21 <sup>3</sup>
$B_1B_2A_1$	2.21.	32	60	2.21	$B_1^5$	.1 <sup>5</sup> .	32	1	.1 <sup>5</sup>
$D_3A_1$	2..3	48	20	2.21					
$D_2B_1A_1$	2.1.2	16	60	2.1 <sup>3</sup>					
$B_1^3A_1$	2.1 <sup>3</sup> .	16	20	2.1 <sup>3</sup>					

**Table 2.** Reflection subgroups of  $W(D_6)$ .

Type of $R$	$\lambda$	$ R $	$  R  $	Class	Type of $R$	$\lambda$	$ R $	$  R  $	Class
$\emptyset$	$1^6.$	1	1	$1^6.$	$A_4$	51.	120	96	51.
$A_1$	$21^4.$	2	30	$21^4.$	$D_4$	$1^2.4$	192	15	$1^2.31$
$D_2$	$1^4.2$	4	15	$1^4.1^2$	$D_2^2$	$1^2.2^2$	16	45	$1^2.1^4$
$A_1^2$	$2^2 1^2.$	4	180	$2^2 1^2.$	$D_2 A_3$	4.2	96	120	$4.1^2$
$A_2$	$31^3.$	6	80	$31^3.$	$D_3 A_2$	3.3	144	80	3.21
$A_1^3$	$2^3.+$	8	60	$2^3.+$	$D_4 A_1$	2.4	384	30	2.31
$A_1^3$	$2^3.-$	8	60	$2^3.-$	$D_2^2 A_1$	$2.2^2$	32	90	$2.1^4$
$D_2 A_1$	$21^2.2$	8	180	$21^2.1^2$	$A_5$	6.+	720	16	6.+
$A_1 A_2$	321.	12	480	321.	$A_5$	6.-	720	16	6.-
$D_3$	$1^3.3$	24	20	$1^3.21$	$D_5$	1.5	1920	6	1.41
$A_3$	$41^2.$	24	120	$41^2.$	$D_2 D_3$	1.32	96	60	$1.21^3$
$D_2 A_1^2$	$2^2.2$	16	180	$2^2.1^2$	$D_6$	.6	23040	1	.51
$D_2 A_2$	31.2	24	240	$31.1^2$	$D_2 D_4$	.42	768	15	.31 <sup>3</sup>
$A_2^2$	33.	36	160	33.	$D_3^2$	.3 <sup>2</sup>	576	10	.2 <sup>2</sup> 1 <sup>2</sup>
$D_3 A_1$	21.3	48	120	21.21	$D_2^3$	.2 <sup>3</sup>	64	15	.1 <sup>6</sup>
$A_1 A_3$	42.+	48	120	42.+					
$A_1 A_3$	42.-	48	120	42.-					

**Table 3.** Reflection subgroups of  $W(E_6)$ .

Type of $R$	$ R $	$  R  $	Class	Type of $R$	$ R $	$  R  $	Class
$\emptyset$	1	1	1	$D_4$	192	45	$D_4$
$A_1$	2	36	$A_1$	$A_1^4$	16	135	$4A_1$
$A_1^2$	4	270	$2A_1$	$A_1 A_2^2$	72	360	$2A_2 + A_1$
$A_2$	6	120	$A_2$	$A_1 A_4$	240	216	$A_4 + A_1$
$A_1^3$	8	540	$3A_1$	$A_5$	720	36	$A_5$
$A_1 A_2$	12	720	$A_2 + A_1$	$D_5$	1920	27	$D_5$
$A_3$	24	270	$A_3$	$A_1^2 A_3$	96	270	$A_3 + 2A_1$
$A_1^2 A_2$	24	1080	$A_2 + 2A_1$	$E_6$	51840	1	$E_6$
$A_2^2$	36	120	$2A_2$	$A_1 A_5$	1440	36	$A_5 + A_1$
$A_1 A_3$	48	540	$A_3 + A_1$	$A_2^3$	216	40	$3A_2$
$A_4$	120	216	$A_4$				

**Table 4.** Reflection subgroups of  $W(E_7)$ .

Type of $R$	$ R $	$  R  $	Class
$\emptyset$	1	1	1
$A_1$	2	63	$A_1$
$A_1^2$	4	945	$2A_1$
$A_2$	6	336	$A_2$
$A_1^3$	8	315	$3A_1'$
$A_1^3$	8	3780	$3A_1''$
$A_1A_2$	12	5040	$A_2 + A_1$
$A_3$	24	1260	$A_3$
$A_1^4$	16	3780	$4A_1'$
$A_1^2A_2$	24	15120	$A_2 + 2A_1$
$A_2^2$	36	3360	$2A_2$
$A_1A_3$	48	1260	$A_3 + A_1'$
$A_1A_3$	48	7560	$A_3 + A_1''$
$A_4$	120	2016	$A_4$
$D_4$	192	315	$D_4$
$A_1^4$	16	945	$4A_1''$
$A_1^3A_2$	48	5040	$A_2 + 3A_1$
$A_1A_2^2$	72	10080	$2A_2 + A_1$
$A_1^2A_3$	96	7560	$A_3 + 2A_1'$
$A_2A_3$	144	5040	$A_3 + A_2$
$A_1A_4$	240	6048	$A_4 + A_1$
$A_1D_4$	384	945	$D_4 + A_1$
$A_1^5$	32	2835	$5A_1$

Type of $R$	$ R $	$  R  $	Class
$A_5$	720	336	$A_5'$
$A_5$	720	1008	$A_5''$
$D_5$	1920	378	$D_5$
$A_1^2A_3$	96	3780	$A_3 + 2A_1''$
$A_1A_2A_3$	288	5040	$A_3 + A_2 + A_1$
$A_2A_4$	720	2016	$A_4 + A_2$
$A_1A_5$	1440	1008	$A_5 + A_1'$
$A_1D_5$	3840	378	$D_5 + A_1$
$A_1^3A_3$	192	3780	$A_3 + 3A_1$
$A_6$	5040	288	$A_6$
$D_6$	23040	63	$D_6$
$A_1^2D_4$	768	945	$D_4 + 2A_1$
$A_3^2$	576	630	$D_4(a_1) + 2A_1$
$A_1^6$	64	945	$6A_1$
$E_6$	51840	28	$E_6$
$A_1A_5$	1440	1008	$A_5 + A_1''$
$A_2^3$	216	1120	$3A_2$
$E_7$	2903040	1	$E_7$
$A_1D_6$	46080	63	$D_6 + A_1$
$A_7$	40320	36	$A_7$
$A_2A_5$	4320	336	$A_5 + A_2$
$A_1A_3^2$	1152	630	$2A_3 + A_1$
$A_1^3D_4$	1536	315	$D_4 + 3A_1$
$A_1^7$	128	135	$7A_1$

**Table 5.** Reflection subgroups of  $W(E_8)$ .

Type of $R$	$ R $	$  R  $	Class	Type of $R$	$ R $	$  R  $	Class
$\emptyset$	1	1	1	$E_6$	51840	1120	$E_6$
$A_1$	2	120	$A_1$	$A_1 A_5$	1440	40320	$A_5 + A'_1$
$A_1^2$	4	3780	$2A_1$	$A_2^3$	216	44800	$3A_2$
$A_2$	6	1120	$A_2$	$A_1 A_2 A_4$	1440	241920	$A_4 + A_2 + 1$
$A_1^3$	8	37800	$3A_1$	$A_3 A_4$	2880	120960	$A_4 + A_3$
$A_1 A_2$	12	40320	$A_2 + A_1$	$A_1 A_6$	10080	34560	$A_6 + A_1$
$A_3$	24	7560	$A_3$	$A_2 D_5$	11520	30240	$D_5 + A_2$
$A_1^4$	16	113400	$4A_1''$	$A_1^2 A_2 A_3$	576	302400	$A_3 + A_2 + 2A_1$
$A_1^2 A_2$	24	302400	$A_2 + 2A_1$	$A_7$	40320	8640	$A_7''$
$A_2^2$	36	67200	$2A_2$	$A_1 E_6$	103680	3360	$E_6 + A_1$
$A_1 A_3$	48	151200	$A_3 + A_1$	$A_1^2 A_5$	2880	120960	$A_5 + 2A_1$
$A_4$	120	24192	$A_4$	$A_1 A_2^3$	432	134400	$3A_2 + A_1$
$D_4$	192	3150	$D_4$	$D_7$	322560	1080	$D_7$
$A_1^4$	16	9450	$4A_1'$	$A_1^2 D_5$	7680	22680	$D_5 + 2A_1$
$A_1^3 A_2$	48	604800	$A_2 + 3A_1$	$A_3 D_4$	4608	37800	$D_4 + A_3$
$A_1 A_2^2$	72	403200	$2A_2 + A_1$	$A_1^4 A_3$	384	113400	$A_3 + 4A_1$
$A_1^2 A_3$	96	453600	$A_3 + 2A_1''$	$E_7$	2903040	120	$E_7$
$A_2 A_3$	144	302400	$A_3 + A_2$	$A_1 D_6$	46080	7560	$D_6 + A_1$
$A_1 A_4$	240	241920	$A_4 + A_1$	$A_7$	40320	4320	$A_7'$
$A_1 D_4$	384	37800	$D_4 + A_1$	$A_2 A_5$	4320	40320	$A_5 + A_2$
$A_1^5$	32	113400	$5A_1$	$A_1 A_2^3$	1152	75600	$2A_3 + A_1$
$A_5$	720	40320	$A_5$	$A_1^3 D_4$	1536	37800	$D_4 + 3A_1$
$D_5$	1920	7560	$D_5$	$A_1^7$	128	16200	$7A_1$
$A_1^2 A_3$	96	75600	$A_3 + 2A_1'$	$E_8$	696729600	1	$E_8$
$A_1^2 A_2^2$	144	604800	$2A_2 + 2A_1$	$D_8$	5160960	135	$D_8$
$A_1 A_2 A_3$	288	604800	$A_3 + A_2 + A_1$	$A_8$	362880	960	$A_8$
$A_1^2 A_4$	480	362880	$A_4 + 2A_1$	$A_1 A_7$	80640	4320	$A_7 + A_1$
$A_3^2$	576	151200	$2A_3''$	$A_1 A_2 A_5$	8640	40320	$A_5 + A_2 + A_1$
$A_2 A_4$	720	241920	$A_4 + A_2$	$A_4^2$	14400	12096	$2A_4$
$A_2 D_4$	1152	50400	$D_4 + A_2$	$A_3 D_5$	46080	7560	$A_7 + A_1$
$A_1^4 A_2$	96	151200	$A_2 + 4A_1$	$A_2 E_6$	311040	1120	$E_6 + A_2$
$A_1 A_5$	1440	120960	$A_5 + A_1''$	$A_1 E_7$	5806080	120	$E_7 + A_1$
$A_1 D_5$	3840	45360	$D_5 + A_1$	$A_1^2 D_6$	92160	3780	$D_6 + 2A_1$
$A_1^3 A_3$	192	453600	$A_3 + 3A_1$	$D_4^2$	36864	1575	$2D_4$
$A_6$	5040	34560	$A_6$	$A_1^2 A_3^2$	2304	37800	$2A_3 + 2A_1$
$D_6$	23040	3780	$D_6$	$A_2^4$	1296	11200	$4A_2$
$A_1^2 D_4$	768	56700	$D_4 + 2A_1$	$A_1^4 D_4$	3072	9450	$D_4 + 4A_1$
$A_3^2$	576	37800	$2A_3'$	$A_1^8$	256	2025	$8A_1$
$A_1^6$	64	56700	$6A_1$				

**Table 6.** Reflection subgroups of  $W(F_4)$ .

Type of $R$	$ R $	$  R  $	Class	Type of $R$	$ R $	$  R  $	Class
$\emptyset$	1	1	1	$F_4$	1152	1	$F_4$
$A_1$	2	12	$A_1$	$B_4$	384	3	$B_4$
$\tilde{A}_1$	2	12	$\tilde{A}_1$	$C_4$	384	3	$B_4$
$A_1\tilde{A}_1$	4	72	$A_1 + \tilde{A}_1$	$\tilde{D}_4$	192	1	$C_3 + A_1$
$A_2$	6	16	$A_2$	$D_4$	192	1	$D_4$
$\tilde{A}_2$	6	16	$\tilde{A}_2$	$\tilde{A}_1B_3$	96	12	$D_4$
$B_2$	8	18	$B_2$	$A_1C_3$	96	12	$C_3 + A_1$
$\tilde{A}_1^2$	4	18	$2A_1$	$B_2^2$	64	9	$D_4(a_1)$
$A_1^2$	4	18	$2A_1$	$A_1\tilde{A}_3$	48	12	$A_3 + \tilde{A}_1$
$A_2\tilde{A}_1$	12	48	$A_2 + \tilde{A}_1$	$A_3\tilde{A}_1$	48	12	$A_3 + \tilde{A}_1$
$A_1\tilde{A}_2$	12	48	$\tilde{A}_2 + A_1$	$A_2\tilde{A}_2$	36	16	$A_2 + \tilde{A}_2$
$B_3$	48	12	$B_3$	$\tilde{A}_1^2B_2$	32	18	$A_3 + \tilde{A}_1$
$A_3$	24	12	$A_3$	$A_1^2B_2$	32	18	$A_3 + \tilde{A}_1$
$\tilde{A}_1B_2$	16	36	$A_3$	$A_1^2\tilde{A}_1^2$	16	18	$4A_1$
$A_1^2\tilde{A}_1$	8	36	$2A_1 + \tilde{A}_1$	$\tilde{A}_1^4$	16	3	$4A_1$
$\tilde{A}_1^3$	8	12	$2A_1 + \tilde{A}_1$	$A_1^4$	16	3	$4A_1$
$C_3$	48	12	$C_3$				
$\tilde{A}_3$	24	12	$B_2 + A_1$				
$A_1B_2$	16	36	$B_2 + A_1$				
$A_1\tilde{A}_1^2$	8	36	$3A_1$				
$A_1^3$	8	12	$3A_1$				

**Table 7.** Reflection subgroups of  $W(G_2)$ .

Type of $R$	$ R $	$  R  $	Class
$\emptyset$	1	1	1
$A_1$	2	3	$A_1$
$\tilde{A}_1$	2	3	$\tilde{A}_1$
$G_2$	12	1	$G_2$
$\tilde{A}_2$	6	1	$A_2$
$A_1\tilde{A}_1$	4	3	$A_1 + \tilde{A}_1$
$A_2$	6	1	$A_2$

**Table 8.** Reflection subgroups of  $W(H_3)$ .

Type of $R$	$ R $	$  R  $	Class
$\emptyset$	1	1	1
$A_1$	2	15	2
$A_1^2$	4	15	4
$A_2$	6	10	5
$I_2(5)$	10	6	3
$H_3$	120	1	6
$A_1^3$	8	5	10

**Table 9.** Reflection subgroups of  $W(H_4)$ .

Type of $R$	$ R $	$  R  $	Class	Type of $R$	$ R $	$  R  $	Class
$\emptyset$	1	1	1	$H_3$	120	60	6
$A_1$	2	60	2	$A_1^3$	8	300	20
$A_1^2$	4	450	4	$H_4$	14400	1	11
$A_2$	6	200	5	$H_3A_1$	240	60	21
$I_2(5)$	10	72	3	$I_2(5)^2$	100	36	26
$A_1A_2$	12	600	8	$A_4$	120	60	27
$I_2(5)A_1$	20	360	7	$A_2^2$	36	100	32
$A_3$	24	300	9	$D_4$	192	25	25
				$A_1^4$	16	75	34



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