ON THE QUIVER PRESENTATION OF THE DESCENT ALGEBRA OF THE SYMMETRIC GROUP

MARCUS BISHOP AND GÖTZ PFEIFFER

Abstract. We describe a presentation for the descent algebra of the symmetric group $\mathfrak{S}_n$ as a quiver with relations. This presentation arises from a new construction of the descent algebra as a homomorphic image of an algebra of forests of binary trees which can be identified with a subspace of the free Lie algebra. In this setting, we provide a new short proof of the known fact that the quiver of the descent algebra of $\mathfrak{S}_n$ is given by restricted partition refinement. Moreover, we describe certain families of relations and conjecture that for fixed $n \in \mathbb{N}$, the finite set of relations from these families that are relevant for the descent algebra of $\mathfrak{S}_n$ generates the ideal of relations, and hence yields an explicit presentation by generators and relations of the algebra.

1. Introduction

Let (W, S) be a finite Coxeter system and let k be a field of characteristic zero. For all $J \subseteq S$ we denote the parabolic subgroup $\langle J \rangle$ of W by W_J and the set of minimal length left coset representatives of W_J in W by X_J. In 1976 Solomon proved [18] that the elements $x_J = \sum_{x \in X_J} x \in kW$ for all $J \subseteq S$ satisfy

$$x_J x_K = \sum_{L \subseteq S} c_{JKL} x_L \tag{1}$$

for certain integers c_{JKL} with $J, K, L \subseteq S$. This implies that the linear span $\langle x_J \mid J \subseteq S \rangle$ is a *subalgebra* of kW. This algebra is called the *descent algebra* of W and is denoted by $\Sigma(W)$.

Solomon shows [18] that the structure constants c_{JKL} in (1) are the same constants appearing in the Mackey formula for the product of the permutation characters $\mathsf{Ind}_{W_J}^W 1$ and $\mathsf{Ind}_{W_K}^W 1$ in terms of the characters $\mathsf{Ind}_{W_L}^W 1$ for all $L \subseteq S$. Therefore the map $\theta : \Sigma(W) \to k\,\mathsf{Irr}(W)$ given by $x_J \mapsto \mathsf{Ind}_{W_J}^W 1$ for all $J \subseteq S$ is a homomorphism of k-algebras, where $k\,\mathsf{Irr}(W)$ is the character ring of W over k. Solomon also shows that $\ker\theta$ is the radical of $\Sigma(W)$.

We identify $k\,\mathsf{Irr}(W)$ with the ring k^m under pointwise addition and multiplication, where m is the number of conjugacy classes in W. Then the map θ above presents the semisimple algebra $\Sigma(W) / \mathsf{Rad}\,\Sigma(W)$ as a subalgebra of k^m. Since k^m is commutative, we conclude that the simple $\Sigma(W)$-modules are all one-dimensional over k so that $\Sigma(W)$ is a basic algebra and therefore has an quiver presentation. See [1] for more information about basic algebras and quivers. The preceding discussion also shows that we can assume k is the field $\mathbb{Q}$ of rational numbers because the permutation characters $\mathsf{Ind}_{W_J}^W 1$ take values in $\mathbb{Z}$ for all $J \subseteq S$.

2000 *Mathematics Subject Classification.* Primary 16G20; Secondary 20F55.
Key words and phrases. Descent algebra, symmetric group, quiver, presentation.

The aim of this paper is to calculate and study the quiver presentation of $\Sigma(W)$ when W is the symmetric group $\mathfrak{S}_n$ of degree $n \geqslant 0$. An elementary proof of equation (1) in this case was given by Atkinson [2] in 1986. The Coxeter generating set of $\mathfrak{S}_n$ is $S = \{1, 2, \dots, n-1\}$ where we identify each $s \in S$ with the transposition exchanging the points s with $s+1$. In this situation the set X_J has a description in terms of the graphs of the elements of $\mathfrak{S}_n$. Here we regard $w \in \mathfrak{S}_n$ as a function from $\{1, 2, \dots, n\}$ to itself and the graph of w as the set of points $\{(i, i.w) \mid 1 \leqslant i \leqslant n\}$. Then X_J is the set of all $w \in \mathfrak{S}_n$ for which $i.w > (i+1).w$ for all $i \in J$, or in other words, the graph of w is *descending* at all points in J. The name *descent algebra* derives from this interpretation.

The algebra $\Sigma(\mathfrak{S}_n)$ plays a major role in the book by Blessenohl and Schocker [9] where the authors study the character theory of $\mathfrak{S}_n$ through an extension of the map θ above to $k\mathfrak{S}_n$. As in [9], this article takes the point of view of studying $\Sigma(\mathfrak{S}_n)$ for all $n \geqslant 0$ simultaneously by uniting objects indexed by n into a single object beginning in §7. The industry of studying $\Sigma(\mathfrak{S}_n)$ through its quiver presentation begins in 1989 with Garsia and Reutenauer's description [12] of the quiver of $\Sigma(\mathfrak{S}_n)$. We derive this quiver in §7 using an algebra $k\mathcal{L}_n$ that we describe below. Garsia and Reutenauer also calculate the Cartan invariants and the projective indecomposable $\Sigma(\mathfrak{S}_n)$-modules. Aktinson [3] derives these using elementary methods.

Bergeron and Bergeron [4, 6] partially describe the quiver of $\Sigma(W)$ for W of type B_n in 1992 with their calculation of the idempotents of $\Sigma(W)$, which correspond with the vertices of the quiver. The full quiver in type B_n was calculated by Saliola [15] in 2008 using hyperplane arrangements.

In a somewhat different direction, but amounting to essentially the same information as a quiver presentation, the *module structure* of $\Sigma(\mathfrak{S}_n)$ was calculated [7, 8] and later expanded by Schocker [17], where he showed that articles [7] and [8] essentially calculate the quiver of $\Sigma(\mathfrak{S}_n)$. One component of the module structure of $\Sigma(W)$ is the length of its Loewy series, which was calculated for W of type D_n for n odd by Saliola [16] in 2010 after the calculation by Bonnafé and Pfeiffer in 2008 [10] for the remaining finite irreducible Coxeter groups.

The first step towards the calculation of the quiver for arbitrary Coxeter groups lies in Bergeron, Bergeron, Howlett, and Taylor's calculation [5] of a basis of idempotents of $\Sigma(W)$ for any Coxeter group W, since these idempotents serve as the vertices of the quiver. Pfeiffer's article [14] builds on the idempotent construction above and shows how one can construct the quiver and the relations for the presentation of $\Sigma(W)$. Since Pfeiffer's construction provides the basis for this article, we briefly summarize it in the following theorem.

Theorem 1. *Let (W, S) be a finite Coxeter system and denote by $\Sigma(W)$ its descent algebra. Then there exist*

- *a category $\mathcal{A}$*
- *an action of the free monoid S^* on $\mathcal{A}$ that partitions $\mathcal{A}$ into orbits*
- *subsets Λ and $\mathcal{E}$ of the set $\mathcal{X}$ of orbits of $\mathcal{A}$*
- *a linear map $\Delta\colon k\mathcal{A} \to k\mathcal{P}$ (where $\mathcal{P}$ is the power set of S)*

such that

- *$k\mathcal{X}$ is a subalgebra of $k\mathcal{A}$ (where we identify the orbit an element of $\mathcal{A}$ with the sum of its elements in $k\mathcal{A}$)*
- *Λ is a complete set of pairwise orthogonal primitive idempotents of $k\mathcal{X}$*
- *$\lambda\,(k\mathcal{X})\,\lambda' \cap \mathcal{X}$ is a basis of the subspace $\lambda\,(k\mathcal{X})\,\lambda'$ for all $\lambda, \lambda' \in \Lambda$*

- *The pair* $(\mathsf{Q}, \ker\Delta)$ *is a quiver presentation of* $\Sigma(W)^{\mathrm{op}}$ *where* Q *is the quiver with vertices* Λ *and edges* $\mathcal{E}$.

We briefly repeat the definitions of the devices introduced in Theorem 1 needed in this article. The category

$$\mathcal{A} = \left\{ (J; s_1, s_2, \ldots, s_l) \;\middle|\; \{s_1, s_2, \ldots, s_l\} \subseteq J \subseteq S \text{ with } s_1, s_2, \ldots, s_l \text{ distinct} \right\}$$

has partial product $\circ : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ defined by

$$(J; s_1, s_2, \ldots, s_l) \circ (K; t_1, t_2, \ldots, t_m) = (J; s_1, s_2, \ldots, s_l, t_1, t_2, \ldots, t_m)$$

if $K = J \setminus \{s_1, s_2, \ldots, s_l\}$. The action of S^* on $\mathcal{A}$ is given by

$$(J; s_1, s_2, \ldots, s_l).t = (J^{\omega}; s_1^{\omega}, s_2^{\omega}, \ldots, s_l^{\omega}) \tag{2}$$

for $t \in S$ and $(J; s_1, s_2, \ldots, s_l) \in \mathcal{A}$ where $\omega = w_J w_{J\cup\{t\}}$ and w_J and $w_{J\cup\{t\}}$ are the longest elements in the parabolic subgroups W_J and $W_{J\cup\{t\}}$ respectively. The superscripts in (2) denote conjugation, so for example $s_1^{\omega} = \omega^{-1} s_1 \omega$. The difference operator δ on $\mathcal{A}$ is defined by $\delta(a) = b - b.s_1$ for $a = (J; s_1, s_2, \ldots, s_l)$ where $b = (J \setminus \{s_1\}; s_2, \ldots, s_l)$ and Δ is defined by iterating δ as many times as possible, so if $a \in \mathcal{A}$ is as above, then $\Delta(a) = \delta^l(a)$. Finally, Λ is the set of orbits of elements of the form $(J;)$ and $\mathcal{E}$ is any maximal linearly independent set of orbits of elements of the form $(J; s_1)$.

Once the quiver provided by Theorem 1 has been identified, the greatest difficulty is in calculating the relations of the presentation, although in principle, this amounts only to transferring $\ker\Delta$ to kQ. Pfeiffer [13] has done this with his explicit quiver presentations of the descent algebras of the Coxeter groups of exceptional and non-crystallographic type. Other than these calculations, no quiver presentations of descent algebras are known, and in contrast with the finite calculation in [13], this paper deals with the calculation of presentations of the algebras in the infinite family $\{\Sigma(\mathfrak{S}_n) \mid n \geqslant 0\}$.

The following is an outline of this paper. The algebras and maps introduced in the outline are shown in the following diagram.

$$\begin{tikzcd}
kQ_n \arrow[r, "\iota"] & k\mathcal{L}_n \arrow[r] \arrow[d, "E"] & kL_n \arrow[d, "E"] \arrow[dr, "\Delta"] & \\
& k\mathcal{M}_n \arrow[r] & kM_n \arrow[r, "\pi"] & k\mathbb{N}^*
\end{tikzcd}$$

To calculate a presentation of $\Sigma(W)$ in the case that $W = \mathfrak{S}_n$ we first develop a simpler description of $\mathcal{A}$. Namely, we show in §4 that in this case each element of $\mathcal{A}$ can be represented as a sequence of binary trees, or a *forest*. The category L_n in the diagram above is the category of forests corresponding with elements of $\mathcal{A}$. The definition and basic properties of forests are the subject of §3. We show in §5 that the monoid action of S^* on L_n amounts simply to rearrangement of the trees of a forest, so the S^*-orbit of an element of $\mathcal{A}$ corresponds with the sum of all rearrangements of the corresponding forest. This action yields the subcategory $\mathcal{L}_n$ of L_n corresponding with $\mathcal{X}$ in Theorem 1. We show in §6 that the map Δ also has a simple description when we represent the elements of $\mathcal{A}$ as forests. Specifically, we introduce categories M_n and $\mathcal{M}_n$ analogous to L_n and $\mathcal{L}_n$ in §9 and show that Δ factors through kM_n in §10. This is accomplished by showing that applying Δ amounts to applying a natural map $E : kL_n \to kM_n$ followed by replacing the nodes of a tree with the Lie bracket in the free associative k-algebra $\mathsf{k}\mathbb{N}^*$. The latter map is denoted by π in the diagram above. This allows us to identify $\Sigma(\mathfrak{S}_n)$ with a quotient of $\mathsf{k}\mathcal{L}_n$ in Theorem 7. We introduce a quiver Q_n in §7 and show in §8 that the path algebra of Q_n can be embedded into the algebra $\mathsf{k}\mathcal{L}_n$ of forest classes through the injective homomorphism ι in the diagram above. We also show in §11 that Q_n is the ordinary quiver of $\Sigma(\mathfrak{S}_n)$. This means that $\Sigma(\mathfrak{S}_n)$ can be identified with a quotient of the path algebra of Q_n by an ideal that can be explicitly calculated. Finally, we present a conjecture in §12 that lists the relations explicitly and we calculate the presentation of $\Sigma(\mathfrak{S}_8)$ in §13, thus verifying the conjecture in this particular example.

2. Compositions, Partitions, and Rearrangement

Much of the charm of the theory developed in this paper stems from the reduction of complicated combinatorial operations to the simpler operation of rearrangement, which is the subject of this section. We denote the free monoid on a set Ω by Ω^*. This is the set of all formal products $x_1 x_2 \cdots x_j$ where $x_i \in \Omega$ for all $1 \leqslant i \leqslant j$. The binary operation on Ω^* is not denoted. In this paper, an important instance of this construction occurs when Ω is the set $\mathbb{N}$ of natural numbers, which does *not* include 0. The elements of $\mathbb{N}^*$ are called *compositions* and the numbers x_i in a composition $x_1 x_2 \cdots x_j$ are called its *parts*.

The symmetric group $\mathfrak{S}_j$ acts on compositions with j parts by

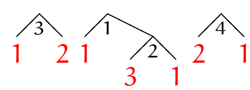
$$(x_1 x_2 \cdots x_j).\pi = x_{1.\pi^{-1}} x_{2.\pi^{-1}} \cdots x_{j.\pi^{-1}}$$

for $\pi \in \mathfrak{S}_j$. This action is called the *Pólya action*. The orbits of the Pólya action on $\mathbb{N}^*$ are called *partitions*. We represent a partition by any of its representatives when this causes no confusion.

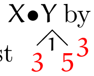
3. Trees and Forests

A *labeled forest* is a sequence of binary trees whose leaves are natural numbers and whose (inner) nodes are labeled by natural numbers in such a way that the label of every node is greater than that of its parent if it has one, and each number $1, 2, \ldots, l$ is the label of exactly one node, where l is the number of nodes in the sequence. For example

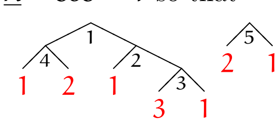
(3)



is labeled forest. Let Y be a labeled forest. The sequence of leaves of Y is called its *foliage* and is denoted $\underline{Y}$. The sum of the leaves of a tree is called its *value*. The sequence of values of the trees of Y is called its *squash* and is denoted $\overline{Y}$. The number of nodes in Y is called its *length* and is denoted $\ell(Y)$. For example, if Y is the forest shown in (3) then $\underline{Y} = 1213121$ and $\overline{Y} = 353$ while $\ell(Y) = 4$.

Whenever two forests X and Y satisfy $\underline{X} = \overline{Y}$ we define a product $X \bullet Y$ by replacing the leaves of X with the trees of Y. For example, if X is the forest  and Y is the forest shown in (3) then $\underline{X} = 353 = \overline{Y}$ so that

(4)



is the product $X \bullet Y$. Note that the node labels of Y must be incremented by $\ell(X)$ to ensure that the product will also be a labeled forest.

All the definitions above can be made mathematically precise by defining a labeled forest to be an element of the free monoid on the set

$$T = \mathbb{N} \cup \left\{ (X_1, i, X_2) \;\middle|\; i \in \mathbb{N} \text{ and } X_1, X_2 \in T \right\}.$$

Then for example, one defines the squash of an element X of T with only finitely many nodes by the formula

$$\overline{X} = \begin{cases} X & \text{if } X \in \mathbb{N} \\ \overline{X_1} + \overline{X_2} & \text{if } X = (X_1, i, X_2) \end{cases}$$

and extends this definition to the free monoid by $\overline{X_1 X_2 \cdots X_j} = \overline{X_1}\;\overline{X_2} \cdots \overline{X_j}$ where $X_1, X_2, \ldots, X_j \in T$. The other functions above can be similarly defined.

Lemma 2. *A labeled forest of length at least one can be uniquely factorized as a product of labeled forests of length one.*

Proof. Suppose that $X = X_1 X_2 \cdots X_j$ is a labeled forest, where $X_1, X_2, \ldots, X_j$ are trees. Note that since 1 is the smallest node label of X, it must be the label of one of the trees $X_1, X_2, \ldots, X_j$, say X_i. This means that $X_i = \overset{\overset{1}{\wedge}}{X_{i1}\;X_{i2}}$ for some trees X_{i1} and X_{i2}. Let Y be obtained from $X_1 X_2 \cdots X_{i-1} X_{i1} X_{i2} X_{i+1} \cdots X_{i+1} X_j$ by reducing the node labels by one and write $x_1 x_2 \cdots x_{i-1} x_{i1} x_{i2} x_{i+1} \cdots x_j = \overline{Y}$. Then if

$$X' = x_1 x_2 \cdots x_{i-1} \quad \overset{\overset{1}{\wedge}}{x_{i1}\;x_{i2}} \quad x_{i+1} \cdots x_j$$

we have $X = X' \bullet Y$. Note that X' is the unique forest of length one with squash $\overline{X}$ and foliage $\overline{Y}$. Repeating the procedure with Y in place of X yields the desired factorization by induction. $\square$

For example, the forest in (4) can be factorized as

$$\left({}_{3}\overset{1}{\wedge}{}_{5} \;\; 3 \right) \bullet \left(3 \;\; {}_{1}\overset{1}{\wedge}{}_{4} \;\; 3 \right) \bullet \left(31 \;\; {}_{3}\overset{1}{\wedge}{}_{1} \;\; 3 \right) \bullet \left({}_{1}\overset{1}{\wedge}{}_{2} \;\; 1313 \right) \bullet \left(12131 \;\; {}_{2}\overset{1}{\wedge}{}_{1} \right). \tag{5}$$

The *value* of a forest is the sum of the values of its trees. For the purpose of constructing the quiver presentation of $\Sigma(\mathfrak{S}_n)$ we restrict our attention to the set L_n of forests of value $n \in \mathbb{N} \cup \{0\}$. Then L_n is a *category*, that is, a monoid whose product is only partially defined. Taking $X \bullet Y$ to be zero whenever $\underline{X} \neq \overline{Y}$ makes kL_n into a k-algebra.

4. Equivalence of Forests with Alleys

Recall from §1 that

$$\mathcal{A} = \left\{ (J; s_1, s_2, \ldots, s_l) \;\middle|\; \{s_1, s_2, \ldots, s_l\} \subseteq J \subseteq S \text{ with } s_1, s_2, \ldots, s_l \text{ distinct} \right\}$$

and that the partial product $\circ : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is defined by

$$(J; s_1, s_2, \ldots, s_l) \circ (K; t_1, t_2, \ldots, t_m) = (J; s_1, s_2, \ldots, s_l, t_1, t_2, \ldots, t_m)$$

if $K = J \setminus \{s_1, s_2, \ldots, s_l\}$. The category $\mathcal{A}$ is a combinatorial gadget used to construct quiver presentations of the descent algebras of finite Coxeter groups. The elements of $\mathcal{A}$ are called *alleys*. The number l is called the *length* of the alley $a = (J; s_1, s_2, \ldots, s_l)$ and is denoted by $\ell(a)$. One can also view a as the chain

$$J \supseteq J \setminus \{s_1\} \supseteq J \setminus \{s_1, s_2\} \supseteq \cdots \supseteq J \setminus \{s_1, s_2, \ldots, s_l\} \tag{6}$$

of subsets of $\{1, 2, \ldots, n\}$. Then the product of two alleys corresponds with the concatenation of the corresponding chains whenever the concatenation is also a chain.

Proposition 3. *The category $\mathcal{A}$ associated to the Coxeter group $\mathfrak{S}_n$ is equivalent to L_n through a length-preserving functor.*

Proof. We identify the Coxeter generating set S of $\mathfrak{S}_n$ with the set $\{1, 2, \ldots, n-1\}$. If $J \subseteq S$ with $|J| = n-j$ then we write $S \setminus J = \{t_1, t_2, \ldots, t_{j-1}\}$ where $t_1 < t_2 < \cdots < t_{j-1}$. We put $t_0 = 0$ and $t_j = n$ and let $\varphi(J)$ be the composition $q_1 q_2 \cdots q_j$ where $q_i = t_i - t_{i-1}$. Then φ is a bijection between the subsets of S and the compositions of n.

Let H_{n-1} be the Hasse diagram of the relation $\subseteq$ on the subsets of S. Then H_{n-1} is a quiver with a vertex for every subset of $\{1, 2, \ldots, n-1\}$ and an arrow from J to K if $|K \setminus J| = 1$. Thanks to the description in (6) we can identify $\mathcal{A}$ with the set of paths of H_{n-1}. Note that under this identification the length of an alley equals the length of the corresponding path.

Now consider the quiver H'_n which has a vertex for every composition of n and an edge from p to q if there exists a forest of length one with foliage p and squash q. Thanks to Lemma 2 we can identify L_n with the set of paths of H'_n. Note that under this identification the length of a forest equals the length of the corresponding path.

Next we observe that the vertices in H_{n-1} are in bijection with the vertices of H'_n through φ and that H_{n-1} has an edge from J to K if and only if H'_n has an edge from $\varphi(J)$ to $\varphi(K)$. This means that the quivers H_{n-1} and H'_n are isomorphic as directed graphs so that $\mathcal{A}$ and L_n are equivalent through a length-preserving functor, which we denote by φ in the following sections. $\square$

For example, the alley $(\{1, 2, 3, 4, 5, 6, 7, 9, 10\}; 3, 4, 7, 1, 10)$ corresponds with the path

$$\begin{aligned}\{1, 2, 3, 4, 5, 6, 7, 9, 10\} \to \{1, 2, 4, 5, 6, 7, 9, 10\} \to \{1, 2, 5, 6, 7, 9, 10\} \\ \to \{1, 2, 5, 6, 9, 10\} \to \{2, 5, 6, 9, 10\} \to \{2, 5, 6, 9\}\end{aligned}$$

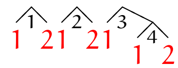
in H_{10}, which in turn corresponds under φ with the path

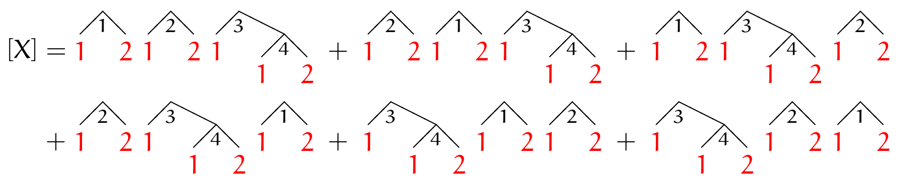
$$83 \to 353 \to 3143 \to 31313 \to 121313 \to 1213121$$

in H'_{11} corresponding with the forest shown in (4) and factorized in (5).

5. Actions and Orbits

If $X = X_1 X_2 \cdots X_j \in L_n$ where $X_1, X_2, \ldots, X_j$ are trees, then $X_1, X_2, \ldots, X_j$ are called the *parts* of X and the Pólya action of $\mathfrak{S}_j$ on compositions with j parts extends to an action on forests with j parts. If $X \in L_n$ is a forest with j parts, then we denote the sum of the elements in the same $\mathfrak{S}_j$-orbit as X by $[X]$. For example, if

X is the forest  then



The set of orbit sums in kL_n is denoted by $\mathcal{L}_n$.

Suppose that $X, Y \in L_n$ are such that $\underline{X} = \overline{Y}$. If X has i parts and Y has j parts, then any element $\sigma \in \mathfrak{S}_i$ induces a permutation $\tau \in \mathfrak{S}_j$ of the leaves of X. Namely, τ is the element satisfying $X.\sigma \bullet Y.\tau = (X \bullet Y).\sigma$. This correspondence is an injective homomorphism when restricted to any subgroup of $\mathfrak{S}_i$ that permutes only parts of X that have the same numbers of leaves. The stabilizer of X in $\mathfrak{S}_i$ is such a subgroup, since it permutes only identical leaves, the parts of positive length having distinct node labels. Therefore the stabilizer of X is isomorphic to a subgroup K of $\mathfrak{S}_j$. Now if H is the stabilizer of Y in $\mathfrak{S}_j$ then

$$[X] \bullet [Y] = \sum_{t=1}^{m} [X \bullet Y.\sigma_t] \in k\mathcal{L}_n$$

where $\sigma_1, \sigma_2, \ldots, \sigma_m$ are representatives of the double cosets of H, K in $\mathfrak{S}_j$. This proves the following proposition.

Proposition 4. $k\mathcal{L}_n$ *is a subalgebra of* kL_n.

Alternately, Proposition 4 follows with Theorem 1 from Proposition 5 below through the equivalence of L_n with $\mathcal{A}$.

Recall from §1 that the free monoid S^* acts on $\mathcal{A}$ by

$$(J; s_1, s_2, \ldots, s_l).t = (J^\omega; s_1^\omega, s_2^\omega, \ldots, s_l^\omega)$$

for $t \in S$ and $(J; s_1, s_2, \ldots, s_l) \in \mathcal{A}$ where $\omega = w_J w_{J \cup \{t\}}$ and w_J and $w_{J \cup \{t\}}$ are the longest elements in the parabolic subgroups W_J and $W_{J \cup \{t\}}$ respectively. When W is the symmetric group we calculate the orbits of this action in the following proposition.

Proposition 5. *The orbits of the Pólya action on* L_n *correspond under the equivalence* φ *in Proposition 3 with the* S^**-orbits on* $\mathcal{A}$*, where* S *is the Coxeter generating set of* $\mathfrak{S}_n$*.*

Proof. Let $a = (J; s_1, s_2, \ldots, s_l) \in \mathcal{A}$ and let $X = \varphi(a) \in L_n$. Let $t_0, t_1, \ldots, t_j$ be as in the proof of Proposition 3. Note that if $t \in J$ then $\omega = w_J w_{J \cup \{t\}} = 1$ so that $a.t = a$. Otherwise assume that $t = t_i$ for some $1 \leqslant i \leqslant j-1$. We claim that $\varphi(a.t_i)$ is obtained from X by exchanging the parts in positions i and $i+1$. From this it will follow that $\varphi(a.S^*) = \varphi(a).\mathfrak{S}_j$.

It is easy to see that conjugation by w_J reverses the elements in the block

$$B_g = \{t_g + 1, t_g + 2, \ldots, t_{g+1} - 1\}$$

for all $0 \leqslant g \leqslant j-1$. Note that including t_i in J joins the blocks B_{i-1} and B_i into the block $B_{i-1} \cup \{t_i\} \cup B_i$. Then since conjugation by $w_{J \cup \{t_i\}}$ again reverses all the blocks, the effect of conjugation by ω is to shift B_{i-1} to the right of B_i while fixing the remaining blocks.

It follows from the definition of φ that if $K \subseteq J$ then $\varphi(K)$ is a refinement of $\varphi(J)$. In other words, if $\varphi(J) = q_1 q_2 \cdots q_j$ then $\varphi(K) = p_1 p_2 \cdots p_j$ where p_i is a composition of q_i for all $1 \leqslant i \leqslant j$. Then conjugating K by ω corresponds under φ with exchanging the compositions p_i and p_{i+1} of $\varphi(K)$. Therefore the path in H'_n corresponding with $\varphi(a.t_i)$ is obtained from the path corresponding with $X = \varphi(a)$ by exchanging the compositions p_i and p_{i+1} of all vertices p. Therefore $\varphi(a.t_i)$ is obtained from X by exchanging the parts in positions i and $i+1$. $\square$

6. Difference Operators

In this section we prove one of the main results of this paper, namely that $\Sigma(\mathfrak{S}_n)$ is isomorphic to a quotient of $k\mathcal{L}_n$. For this purpose we define a difference operator δ on kL_n as follows. Suppose that $X = X_1 X_2 \cdots X_j \in L_n$ where $X_1, X_2, \ldots, X_j$ are trees and X_i is the node labeled 1. Then $X_i = \overset{1}{\widehat{X_{i1}\,X_{i2}}}$ for some trees X_{i1} and X_{i2}. We define $\delta(X)$ to be the element of kL_n obtained from X by replacing X_i with the Lie bracket $X_{i1}X_{i2} - X_{i2}X_{i1}$ and reducing the remaining node labels by one. In terms of the Pólya action, this means that $\delta(X) = Y - Y.i$ where Y is the forest obtained from X by splitting the part $\overset{1}{\widehat{X_{i1}\,X_{i2}}}$ in position i into $X_{i1}X_{i2}$ and reducing the remaining node labels by one.

Recall from §1 that the difference operator δ on $\mathcal{A}$ is defined by $\delta(a) = b - b.s_1$ for all $a = (J; s_1, s_2, \ldots, s_l)$ where $b = (J \setminus \{s_1\}; s_2, \ldots, s_l)$. When W is the symmetric group, this difference operator coincides with the one introduced above in the following sense.

Proposition 6. $\varphi(\delta(a)) = \delta(\varphi(a))$ *for all alleys* $a \in \mathcal{A}$ *associated with* $\mathfrak{S}_n$.

Proof. Let a and b be as above and let $X = X_1 X_2 \cdots X_j = \varphi(a) \in L_n$ where $X_1, X_2, \ldots, X_j$ are trees. The factorization $a = (J; s_1) \circ b$ and the factorization $X = X' \bullet Y$ in Lemma 2 imply that $\varphi(J; s_1) = X'$ and $\varphi(b) = Y$ by unique factorization and length-preserving equivalence.

Now let $t_1, \ldots, t_{j-1}$ be as in Proposition 3. Then

$$\{1, 2, \ldots, n-1\} \setminus (J \setminus \{s_1\}) = \{t_1, t_2, \ldots, t_{i-1}, s_1, t_i, t_{i+1}, \ldots t_{j-1}\}$$

with $t_1 < t_2 < \cdots < t_{i-1} < s_1 < t_i < t_{i+1} < \cdots < t_{j-1}$. Since s_1 is in position i of this list, $\varphi(b.s_1)$ is obtained from $\varphi(b)$ by exchanging the trees in positions i and $i+1$ by the proof of Proposition 5. Thus $\delta(X) = Y - Y.i = \varphi(b - b.s_1) = \varphi(\delta(a))$. $\square$

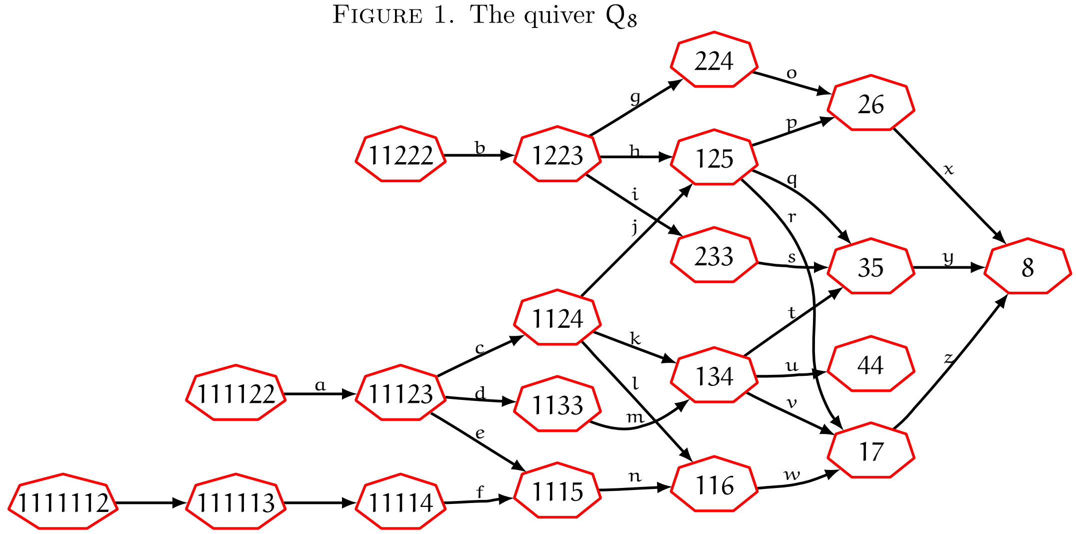
Iterating δ as many times as possible determines another difference operator Δ defined by $\Delta(X) = \delta^{\ell(X)}(X)$ and $\Delta(a) = \delta^{\ell(a)}(a)$ for all forests X and alleys a. Thus, applying Δ to $X \in L_n$ results in a $\mathbb{Z}$-linear combination of compositions of n.

Theorem 7. $\Sigma(\mathfrak{S}_n)$ *is isomorphic to* $k\mathcal{L}_n / \ker \Delta$.

Proof. $k\mathcal{X}/\ker \Delta$ is isomorphic to $\Sigma(\mathfrak{S}_n)$ by Theorem 1 and $k\mathcal{X}$ is isomorphic to $k\mathcal{L}_n$ by Proposition 5. Then $k\mathcal{X} \cong k\mathcal{L}_n$ since the maps Δ on the two algebras coincide under φ by Proposition 6. $\square$

Theorem 7 gives a new construction of $\Sigma(\mathfrak{S}_n)$ as a quotient of $\mathcal{L}_n$, which in turn is a homomorphic image of the path algebra of a quiver, as we show in the following sections.

Figure 1. The quiver Q_8



7. The Quiver

Recall from Lemma 2 that a labeled forest of length at least one can be uniquely factorized as a product of forests of length one. This property fails when we replace L_n with $\mathcal{L}_n$. For example, if we try to factorize $\left[\, 1 \overset{1}{\wedge} \left(1 \overset{2}{\wedge} 2\right) \; 3 \,\right]$ as the product of $\left[\, 1 \overset{1}{\wedge} 3 \; 3 \,\right]$ and $\left[\, 1 \overset{1}{\wedge} 2 \; 12 \,\right]$ we find that the product

$$\left[\, 1 \overset{1}{\wedge} 3 \; 3 \,\right] \bullet \left[\, 1 \overset{1}{\wedge} 2 \; 12 \,\right] = \left[\, 1 \overset{1}{\wedge} \left(1 \overset{2}{\wedge} 2\right) \; 3 \,\right] + \left[\, 1 \overset{1}{\wedge} 3 \; 1 \overset{2}{\wedge} 2 \,\right]$$

has an extra term. This defect in factorization is the subject of §11.

Nonetheless, the success of factorization in L_n suggests representing the algebra $k\mathcal{L}_n$ as a path algebra. Namely, in the factorization of any labeled forest, the foliage of each factor equals the squash of the following factor, so we can regard the factors such a factorization as edges connecting partitions of n.

Let Q_n be the quiver having the partitions of n as vertices and an edge from the vertex p to the vertex q whenever q can be obtained from p by replacing two *distinct* parts with their sum. In other words, Q_n is the Hasse diagram of the partitions of n under restricted partition refinement. The requirement that the parts be distinct will be explained in §10. For example, the quiver Q_8 is shown in Figure 1, omitting the vertices 11111111 and 2222, which are not incident with any edges.

Consider the map $\iota : Q_n \to k\mathcal{L}_n$ given by $\iota(p) = [p]$ if p is a vertex of Q_n and $\iota(e) = \left[\, a \overset{1}{\wedge} b \;\; q_1 q_2 \cdots q_j \,\right]$ if e is the edge going from abq to $(a+b)q$ for some $a, b \in \mathbb{N}$ with $a < b$ and some partition $q = q_1 q_2 \cdots q_j$. Note that ι satisfies $\iota(xy) = \iota(y)\iota(x)$ whenever one of x or y is a vertex and the other is an incident vertex or edge. This proves the following proposition.

Proposition 8. ι *extends to an anti-homomorphism* $\iota : kQ_n \to k\mathcal{L}_n$*.*

We show in Corollary 12 that ι is injective and in Proposition 22 that Q_n is the ordinary quiver of $\Sigma(\mathfrak{S}_n)$. One of the main ingredients in the proof of Proposition 22 is the following lemma.

Lemma 9. *If* e *is an edge of* Q_n *then* $\iota(e) \notin \ker \Delta$.

Proof. Suppose $\iota(e) = \left[\begin{smallmatrix} & 1 & \\ a & & b \end{smallmatrix} \; q_1 q_2 \cdots q_j \right]$ and that $0 \leqslant i \leqslant j$ is such that $q_1 \leqslant q_2 \leqslant \cdots \leqslant q_i \leqslant a < q_{i+1} \leqslant \cdots \leqslant q_j$. Then the term $q_1 q_2 \cdots q_i a b q_{i+1} \cdots q_j$ of $\Delta\left(q_1 q_2 \cdots q_i \; \begin{smallmatrix} & 1 & \\ a & & b \end{smallmatrix} \; q_{i+1} \cdots q_j \right)$ has at most one descending subsequence, namely bq_{i+1}. However, all the terms of $\Delta(\iota(e))$ appearing with negative coefficients have the descending subsequence ba which is different from bq_{i+1} since $a < q_{i+1}$. Thus $\Delta(\iota(e))$ cannot be zero. □

In an effort both to simplify notation and to shift emphasis from the individual groups $\mathfrak{S}_n$ to the family $\bigcup_{n \in \mathbb{N} \cup \{0\}} \mathfrak{S}_n$ of groups, we define

$$Q = \coprod_{n \in \mathbb{N} \cup \{0\}} Q_n \qquad L = \coprod_{n \in \mathbb{N} \cup \{0\}} L_n \qquad k\mathcal{L} = \coprod_{n \in \mathbb{N} \cup \{0\}} k\mathcal{L}_n$$

and regard ι as a map $kQ \to k\mathcal{L}$.

8. The Branch Monoid

Let $\mathcal{B}$ be the set of symbols $\left\langle \begin{smallmatrix} a \\ b \end{smallmatrix} \right|$ for all $a, b \in \mathbb{N}$ with $a < b$. We call the free monoid $\mathcal{B}^*$ the *branch monoid* and we write the element $\left\langle \begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix} \right| \left\langle \begin{smallmatrix} a_2 \\ b_2 \end{smallmatrix} \right| \cdots \left\langle \begin{smallmatrix} a_l \\ b_l \end{smallmatrix} \right|$ of $\mathcal{B}^*$ as $\left\langle \begin{smallmatrix} a_1 & a_2 & \cdots & a_l \\ b_1 & b_2 & \cdots & b_l \end{smallmatrix} \right|$ to simplify notation. The notation is meant to reflect the fact the elements of $\mathcal{B}^*$ can be used to build forests as we now describe.

If X is a forest then let $X.\left\langle \begin{smallmatrix} a \\ b \end{smallmatrix} \right|$ be the sum of all forests that can be obtained from X by replacing a leaf $a + b$ with $\begin{smallmatrix} & l & \\ a & & b \end{smallmatrix}$ where $l = \ell(X) + 1$. If P is a path in Q with source p then let $P.\left\langle \begin{smallmatrix} a \\ b \end{smallmatrix} \right|$ be the path obtained from P by appending the edge $\{abq \to p\}$ on the left if $p = (a + b)\, q$ for some partition q and let $P.\left\langle \begin{smallmatrix} a \\ b \end{smallmatrix} \right| = 0$ otherwise. Then $\mathcal{B}^*$ acts on kL and on kQ by extending the definitions above by linearity. From the definitions we have

$$(P_1 P_2).B = (P_1.B)\, P_2 \qquad \text{and} \qquad (X_1 \bullet X_2).B = X_1 \bullet X_2.B \tag{7}$$

for $P_1, P_2 \in kQ$ and $X_1, X_2 \in kL$ and $B \in \mathcal{B}^*$. If p and $q = q_1 q_2 \cdots q_j$ are as above, then

$$\iota(p).\left\langle \begin{matrix} a \\ b \end{matrix} \right| = [p].\left\langle \begin{matrix} a \\ b \end{matrix} \right| = \left[\begin{matrix} & 1 & \\ a & & b \end{matrix} \quad q_1 q_2 \cdots q_j \right] = \iota\left(p.\left\langle \begin{matrix} a \\ b \end{matrix} \right| \right) \tag{8}$$

whereas both $\iota(p).\left\langle \begin{smallmatrix} a \\ b \end{smallmatrix} \right|$ and $p.\left\langle \begin{smallmatrix} a \\ b \end{smallmatrix} \right|$ are zero if p has no part $a + b$. Now if P is a path in Q with source p, then using (7) and (8) we have

$$\iota(P).B = \iota(pP).B = (\iota(P) \bullet \iota(p)).B = \iota(P) \bullet \iota(p.B) = \iota((p.B)\, P) = \iota(P.B)$$

for all $B \in k\mathcal{B}^*$. This proves the following proposition.

Proposition 10. ι *is a homomorphism of* $k\mathcal{B}^*$*-modules.*

The branch monoid provides a convenient language for specifying paths in Q. Namely, we can uniquely specify any path P as $p.B$ where p is the destination of P and B is an element of $\mathcal{B}^*$. Furthermore, the element B is related to $\iota(P)$ in the way described in the following lemma.

Lemma 11. *Let* $P = p.\left\langle \begin{smallmatrix} a_1 & a_2 & \cdots & a_l \\ b_1 & b_2 & & b_l \end{smallmatrix} \right|$ *be a path in* Q *where* p *is a vertex. Then the node* $\overset{j}{\widehat{Z_1\ Z_2}}$ *of every term of* $\iota(P)$ *satisfies* $\overline{Z_1} = a_j$ *and* $\overline{Z_2} = b_j$ *for all* $1 \leqslant j \leqslant l$.

Proof. This is true by definition if l equals zero or one. Let $P' = p.\left\langle \begin{smallmatrix} a_1 & a_2 & \cdots & a_{l-1} \\ b_1 & b_2 & & b_{l-1} \end{smallmatrix} \right|$ so that $P = P'.\left\langle \begin{smallmatrix} a_l \\ b_l \end{smallmatrix} \right|$ and $\iota(P) = \iota(P').\left\langle \begin{smallmatrix} a_l \\ b_l \end{smallmatrix} \right|$ by Proposition 10. Then $\iota(P)$ is obtained from $\iota(P')$ by replacing a leaf $a_l + b_l$ in every term with $\overset{l}{\widehat{a_l\ b_l}}$. Thus the node labeled l of every term of $\iota(P)$ satisfies the condition in the Lemma, while the other nodes satisfy the condition by induction. □

Corollary 12. *The anti-homomorphism* ι *is injective.*

Proof. By Lemma 11 the images of distinct paths are supported on disjoint subsets of $\mathcal{L}$. □

9. Unlabeled Forests

To compute the kernel of $\Delta : \mathcal{L} \to k\mathbb{N}^*$ it will be helpful to introduce an algebra through which Δ factors. Then the kernel of Δ can be assembled from the kernels of its factors. Let M be the category of *unlabeled forests*, which are simply sequences of binary trees whose leaves are natural numbers. The definitions of the foliage, squash, length, value, and product of unlabeled forests can be easily adapted from the definitions for labeled forests, as can the Pólya action and the action of $k\mathcal{B}^*$ on M. Then

$$M = \coprod_{n \in \mathbb{N} \cup \{0\}} M_n \qquad \text{and} \qquad \mathcal{M} = \coprod_{n \in \mathbb{N} \cup \{0\}} \mathcal{M}_n$$

where M_n is the category of unlabeled forests of value n and $\mathcal{M}$ and $\mathcal{M}_n$ are the categories of Pólya class sums in kM and kM_n.

There is a map $E : L \to M$ given by erasing the node labels of a forest. If X is a labeled forest with j parts, then we denote by α_X the index of the stabilizer of X in $\mathfrak{S}_j$ in the stabilizer of $E(X)$ in $\mathfrak{S}_j$.

Lemma 13. *If* $X \in L$ *then* $E[X] = \alpha_X [E(X)]$.

For example, if $X = \overset{1}{\widehat{1\ 2}}\overset{2}{\widehat{1\ 2}}$ then $[X] = \overset{1}{\widehat{1\ 2}}\overset{2}{\widehat{1\ 2}} + \overset{2}{\widehat{1\ 2}}\overset{1}{\widehat{1\ 2}}$ while $[E(X)] = \widehat{1\ 2}\,\widehat{1\ 2}$ so that $E[X] = 2[E(X)]$.

Recall that the product in L or M of two forests is formed by replacing the leaves in one forest with the trees of the other. Since this process depends on foliage and squash but not node labels, we observe that up to node label erasure, the same products are formed with or without the node labels. This means that E is a functor and the induced map $E : kL \to kM$ is an algebra homomorphism. Then since the restriction of E to the subalgebra $k\mathcal{L}$ has image in $k\mathcal{M}$ by Lemma 13, we have the following result.

Proposition 14. *The map* $E : k\mathcal{L} \to k\mathcal{M}$ *given by erasing node labels is an algebra homomorphism.*

As with labeled forests, the definition of unlabeled forests can be made mathematically precise by defining unlabeled trees to be elements of the free monoid on the set

$$U = \mathbb{N} \cup \left\{ (X_1, X_2) \;\middle|\; X_1, X_2 \in U \right\}.$$

Then for example, the map E can be defined by

$$\mathsf{E}(X) = \begin{cases} X & \text{if } X \in \mathbb{N} \\ (\mathsf{E}(X_1), \mathsf{E}(X_2)) & \text{if } X = (X_1, i, X_2) \end{cases}.$$

10. Alignment

Let $\mathbb{M}$ be the free magma generated by $\mathbb{N}$. The product of two elements X and Y of $\mathbb{M}$ is $\widehat{X\;Y}$. Although we could introduce a symbol for this operation, after several iterations, it becomes more instructive to simply represent an element of $\mathbb{M}$ as a binary tree, that is, as an unlabeled forest with exactly one part.

We define the ideals

$$\mathsf{N} = \left\langle \widehat{X\;Y} + \widehat{Y\;X} \;\middle|\; X, Y \in \mathbb{M} \right\rangle$$
$$\mathsf{J} = \left\langle \widehat{X\;\widehat{Y\;Z}} + \widehat{Y\;\widehat{Z\;X}} + \widehat{Z\;\widehat{X\;Y}} \;\middle|\; X, Y, Z \in \mathbb{M} \right\rangle$$

of $k\mathbb{M}$ and recall that $k\mathbb{M}/(\mathsf{N}+\mathsf{J})$ defines the free Lie algebra over k generated by $\mathbb{N}$.

Since the elements of $\mathbb{M}$ correspond with elements of M that have exactly one part, we can identify arbitrary elements of M with the elements of the free monoid $\mathbb{M}^*$. Under this identification, the category M has, in addition to $\bullet$, another product coming from concatenation in $\mathbb{M}^*$. Let $\mathcal{N}$ and $\mathcal{J}$ be the ideals of $k\mathsf{M}$ with respect to concatenation generated by N and J respectively.

Let $\pi : k\mathbb{M} \to k\mathbb{N}^*$ be defined by $\pi(x) = x$ for $x \in \mathbb{N}$ and $\pi\left(\widehat{X\;Y}\right) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$ for $X, Y \in \mathbb{M}$. Then π extends to a monoid algebra homomorphism $\pi : k\mathsf{M} \to k\mathbb{N}^*$ and the kernel of π is the ideal $\mathcal{N}+\mathcal{J}$ generated by the kernel $\mathsf{N}+\mathsf{J}$ of $\pi : k\mathbb{M} \to k\mathbb{N}^*$. Recall that the map Δ replaces nodes of labeled trees with Lie brackets in the order given by the node labels. The relationship between Δ and π is the following.

Lemma 15. $\Delta = \pi \circ \mathsf{E}$

Proof. Let $X = X_1X_2\cdots X_j \in \mathsf{L}$ where $X_1, X_2, \ldots, X_j$ are trees and suppose that the node labeled 1 is X_i so that $X_i = \overset{1}{\widehat{X_{i1}\,X_{i2}}}$ for some trees X_{i1} and X_{i2}. Then

$$\begin{aligned} \pi(\mathsf{E}(X)) &= \pi(\mathsf{E}(X_1))\cdots\pi(\mathsf{E}(X_i))\cdots\pi(\mathsf{E}(X_j)) \\ &= \pi(\mathsf{E}(X_1))\cdots\pi(\mathsf{E}(X_{i1}X_{i2} - X_{i2}X_{i1}))\cdots\pi(\mathsf{E}(X_j)) \\ &= \pi(\mathsf{E}(X_1\cdots(X_{i1}X_{i2} - X_{i2}X_{i1})\cdots X_j)) \\ &= \pi(\mathsf{E}(\delta(X))). \end{aligned}$$

Now since $\delta(X)$ has shorter length than X, we have $\pi(\mathsf{E}(\delta(X))) = \Delta(\delta(X)) = \Delta(X)$ by induction. $\square$

A forest X is called *aligned* if $\overline{Z_1} < \overline{Z_2}$ for all nodes $\widehat{Z_1 Z_2}$ of X. Since the product of two aligned forests is aligned, the category M^+ of aligned unlabeled forests is a subcategory of M and

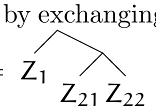
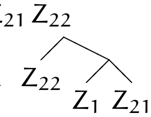
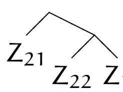
$$\mathsf{M}^+ = \coprod_{n \in \mathbb{N}\cup\{0\}} \mathsf{M}_n^+ \qquad \text{and} \qquad \mathcal{M}^+ = \coprod_{n \in \mathbb{N}\cup\{0\}} \mathcal{M}_n^+$$

where M^+_n is the category of aligned unlabeled forests of value n and $\mathcal{M}^+$ and $\mathcal{M}^+_n$ are the categories of class sums in kM^+ and kM^+_n. We similarly define the categories of aligned labeled trees L^+, L^+_n, $\mathcal{L}^+$, $\mathcal{L}^+_n$. Our first observation about aligned forests is that the image of ι is aligned.

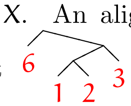
Lemma 16. $\iota(kQ) \subseteq k\mathcal{L}^+$

Proof. We observe that $\iota(e)$ is aligned for each edge e of Q as a result of the requirement $a < b$ in the definition of ι. Then since ι is a homomorphism by Proposition 8 it follows that the images of all elements of kQ under ι are aligned. $\square$

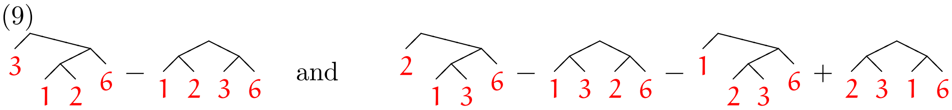
Lemma 17. *If* $X \in M$ *and no node* $\widehat{Z_1 Z_2}$ *of* X *satisfies* $Z_1 = Z_2 \in \mathbb{N}$ *then there exist* $A \in M^+$ *and* $Y \in \mathcal{N} + \mathcal{J}$ *such that* $A = X + Y$.

Proof. If X is aligned, then we can take $A = X$ and $Y = 0$. Otherwise let $Z = \widehat{Z_1 Z_2}$ be a node of X for which $\overline{Z_1} \geqslant \overline{Z_2}$. We define an auxiliary element $X' \in kM$ as follows. If $\overline{Z_1} > \overline{Z_2}$ then let X' be the forest obtained from X by exchanging Z_1 with Z_2 so that $X + X' \in \mathcal{N}$. If $\overline{Z_1} = \overline{Z_2}$ and $\ell(Z_2) > 0$ then $Z =$  for some trees Z_{21} and Z_{22}. Let X' obtained from X by replacing Z with  $+$  so that $X + X' \in \mathcal{J}$. Finally, if $\overline{Z_1} = \overline{Z_2}$ and $\ell(Z_1) > 0$ then we can apply both replacements above to define an element X' such that $X + X' \in \mathcal{N} + \mathcal{J}$.

Observe that each term of X' has fewer nodes $\widehat{U_1 U_2}$ with $U_1 \geqslant U_2$ than X. Then by induction $A' = X' + Y''$ for some $A' \in M^+$ and some $Y'' \in \mathcal{N} + \mathcal{J}$. Taking $A = -A'$ and $Y = -Y' - Y''$ gives the result. $\square$

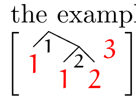
The forest A in Lemma 17 is called an *aligned rendering* of X. An aligned rendering of a forest need not be unique. For example, the forest  has aligned renderings

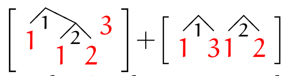
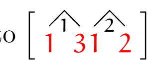
(9)



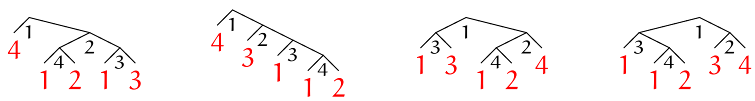
obtained by applying the replacements in Lemma 17 to different nodes.

11. Surjectivity and Proof of the Quiver

Continuing the example at the beginning of §7 we recall that Q was constructed on the basis of unique factorization of labeled forests. However, when mapping the quiver back to the algebra of labeled forests, we replaced the factors in such a factorization with their Pólya classes, which are more useful in light of our interest in $\Sigma(\mathfrak{S}_n)$ but which ruin the factorization, as the example shows. Specifically we associated the path $P = 34.\langle\begin{smallmatrix}1 & 1\\ 3 & 2\end{smallmatrix}|$ to the class  and found that $\iota(P) =$

. Furthermore, applying the same procedure to  results in the same path P, so again the factorization fails. Therefore, we take a closer look at the image of a path under ι and the path associated to a labeled forest.

If a labeled forest has subtrees U and V satisfying $\overline{\mathsf{U}} = \overline{\mathsf{V}}$, then exchanging them results in another labeled forest, provided that the node labels of the parents of U and V, if they exist, are *smaller* than the node labels of U and V, if they exist. We observe that if neither U nor V has a parent, that is, if U and V are *parts* of a forest, then exchanging U and V results in a forest in the same Pólya class. We write $[\mathsf{X}] \sim [\mathsf{Y}]$ for $\mathsf{X}, \mathsf{Y} \in \mathsf{L}$ if $[\mathsf{Y}]$ can be obtained from $[\mathsf{X}]$ by applying a sequence exchanges of subtrees of the same squash to X. Then $\sim$ is an equivalence relation on the set of Pólya classes of labeled forests. Note that if $[\mathsf{X}] \sim [\mathsf{Y}]$ then $[\mathsf{X}]$ is aligned if and only if $[\mathsf{Y}]$ is aligned. For example, the classes of the forests

(10) 

are related by $\sim$.

As in the example at the beginning of this section, we associate a path to an aligned labeled forest through the map $\mathsf{P} : \mathsf{L}^+ \to k\mathsf{Q}$ defined by $\mathsf{P}(\mathsf{X}) = \overline{\mathsf{X}}.\big\langle\begin{smallmatrix} a_1 & a_2 & \cdots & a_l \\ b_1 & b_2 & & b_l \end{smallmatrix}\big|$ for $\mathsf{X} \in \mathsf{L}^+$ where $a_1, b_1, a_2, b_2, \ldots, a_l, b_l \in \mathbb{N}$ are such that the node $\genfrac{}{}{0pt}{}{j}{\mathsf{Z}_1 \;\; \mathsf{Z}_2}$ of X satisfies $\overline{\mathsf{Z}_1} = a_j$ and $\overline{\mathsf{Z}_2} = b_j$ for all $1 \leqslant j \leqslant l$. As the example illustrates, applying P to the terms of the image under ι of a path P results in the same path by Lemma 11, which must therefore be P. We also observe that applying P to forests in the same Pólya class produces the same path. Thus we can define $\mathsf{P}[\mathsf{X}] = \mathsf{P}(\mathsf{X})$. For example, if X is any of the forests in (10) then $\mathsf{P}[\mathsf{X}] = p.\big\langle\begin{smallmatrix} 4 & 3 & 1 & 1 \\ 7 & 4 & 3 & 2 \end{smallmatrix}\big|$ where p is the partition containing the single part eleven.

The map P can also be formulated recursively by putting $\mathsf{P}(\mathsf{X}) = \mathsf{X}$ if $\ell(\mathsf{X}) = 0$ or $\mathsf{P}(\mathsf{X}) = \mathsf{P}(\mathsf{Y})\left(\overline{\mathsf{X}}.\big\langle\begin{smallmatrix} x_{i1} \\ x_{i2} \end{smallmatrix}\big|\right)$ otherwise, where X', Y are as in Lemma 2. Note that $[\mathsf{X}'] = \iota\left(\overline{\mathsf{X}}.\big\langle\begin{smallmatrix} x_{i1} \\ x_{i2} \end{smallmatrix}\big|\right)$ so that ι and P are inverses of one another when restricted to elements of length one. The same is true of elements of length zero. The following lemma deals with the composition $\iota \circ \mathsf{P}$ in general.

Lemma 18. *If* $\mathsf{X} \in \mathsf{L}^+$ *then* $\iota(\mathsf{P}[\mathsf{X}]) = \displaystyle\sum_{[\mathsf{U}] \sim [\mathsf{X}]} [\mathsf{U}]$.

Proof. As mentioned above $\iota(\mathsf{P}[\mathsf{X}]) = [\mathsf{X}]$ if X has length zero or one. Otherwise let Y be as in the definition above. Assuming by induction that $\iota(\mathsf{P}[\mathsf{Y}]) = \sum_{[\mathsf{V}] \sim [\mathsf{Y}]} [\mathsf{V}]$ we have

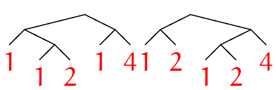
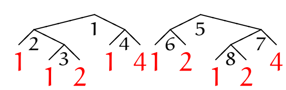
$$\iota(\mathsf{P}[\mathsf{X}]) = \left[\, x_1 x_2 \cdots x_{i-1} \;\; \genfrac{}{}{0pt}{}{1}{x_{i1} \;\; x_{i2}} \;\; x_{i+1} \cdots x_j \,\right] \bullet \sum_{[\mathsf{V}] \sim [\mathsf{Y}]} [\mathsf{V}]\,. \tag{11}$$

Note that all the terms of (11) satisfy $[\mathsf{U}] \sim [\mathsf{X}]$. Conversely, suppose that $[\mathsf{U}]$ is such that $[\mathsf{U}] \sim [\mathsf{X}]$. We can assume that U can be obtained from X by exchanging a single pair of subtrees of the same squash since $\sim$ is the reflexive and transitive closure of the set of all such pairs of forests. If the exchange moves the node labeled 1 then it must exchange it with another part of X since 1 is the smallest node label in X. Then $[X] = [U]$. Otherwise $P[U] = P[V]\left(\overline{X}.\langle\frac{\overline{X_{i1}}}{X_{i2}}|\right)$ for some forest V such that $[V] \sim [Y]$. This shows that $[U]$ is a term of (11). □

In a similar spirit we can define an element $F(X) \in L^+$ such that $E(F(X)) = X$ for all $X \in M^+$. While this can be done by simply labeling the nodes of X in any legitimate way, the labeling provided by F is convenient in the proofs of the following results. If X has length zero, then X is also in L^+ and we can take $F(X) = X$. Otherwise suppose $X = X_1X_2\cdots X_j$ where $X_1, X_2, \ldots, X_j$ are trees. Let i be minimal such that $\ell(X_i) > 0$ and let X_{i1}, X_{i2} be trees such that $X_i = \overset{i}{\widehat{X_{i1}\,X_{i2}}}$. Let Y be obtained from $X_1X_2\cdots X_{i-1}X_{i1}X_{i2}X_{i+1}\cdots X_j$ by reducing all the node labels by one and write $x_1x_2\cdots x_{i1}x_{i2}\cdots x_j = \overline{Y}$. Then defining

$$F(X) = \left(x_1x_2\cdots x_{i-1}\ \overset{1}{\widehat{x_{i1}\ x_{i2}}}\ x_{i+1}\cdots x_j \right) \bullet F(Y)$$

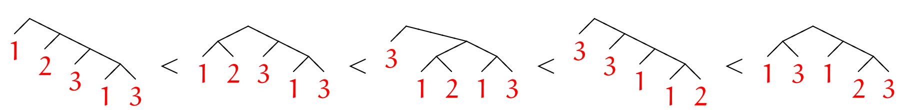
we have $E(F(X)) = X$ by induction. Note that the nodes in $F(X)$ are labeled in prefix order and that the node labels in any part of $F(X)$ are smaller than those in the following part. For example, if

$X =$  then $F(X) =$  .

Next we introduce a total order $<$ on the set of aligned unlabeled trees. Let X and Y be aligned unlabeled trees. If $\ell(X) > 0$ then let X_1, X_2 be trees such that $X = \widehat{X_1X_2}$ and similarly for Y. Then we define $X < Y$ if one of the following conditions holds.

(1) $\overline{X} < \overline{Y}$
(2) $\overline{X} = \overline{Y}$ and $\ell(X) > \ell(Y)$
(3) $\overline{X} = \overline{Y}$ and $\ell(X) = \ell(Y)$ and $X_1 < Y_1$
(4) $\overline{X} = \overline{Y}$ and $\ell(X) = \ell(Y)$ and $X_1 = Y_1$ and $X_2 < Y_2$

Note that in situations (3) and (4) the trees X_1, X_2, Y_1, Y_2 have length shorter than $\ell(X) = \ell(Y)$ and can therefore be compared by induction. For example, the following trees are sorted according to $<$.



The relation $<$ induces the lexicographic order on unlabeled forests, which is also denoted by $<$. This allows us to introduce the notion of a *nondecreasing representative* $X \in M^+$ of its class $[X]$, namely the element whose parts appear in nondecreasing order. The most important property of the nondecreasing representative is given in the following lemma.

Lemma 19. *If* $X \in M^+$ *is nondecreasing then* $[X] < [E(Z)]$ *for all* $[Z] \neq [F(X)]$ *such that* $[Z] \sim [F(X)]$.

Proof. Let $p_1p_2\cdots p_j = \overline{X}$ and let $\left\langle\begin{smallmatrix}a_1\\b_1\end{smallmatrix}\cdots\begin{smallmatrix}a_l\\b_l\end{smallmatrix}\right| \in \mathcal{B}^*$ be such that $\mathsf{F}(X)$ is a term of $p.\left\langle\begin{smallmatrix}a_1\\b_1\end{smallmatrix}\cdots\begin{smallmatrix}a_l\\b_l\end{smallmatrix}\right|$. Then any $Z \in L$ such that $Z \sim \mathsf{F}(X)$ can be assembled from the set

$$p_1, p_2, \ldots, p_j,\ \overset{1}{\widehat{a_1\quad b_1}}, \ldots,\ \overset{l}{\widehat{a_l\quad b_l}}$$

by replacing an element equal to $a_i + b_i$ in the list

$$p_1, p_2, \ldots, p_j, a_1, b_1, \ldots, a_l, b_l \tag{12}$$

with $\overset{i}{\widehat{a_i\quad b_i}}$ for all $1 \leqslant i \leqslant j$. This sequence of replacements can in turn be identified with an injective function $\{1, 2, \ldots, l\} \to \{1, 2, \ldots, j + 2l\}$. Viewing $\mathsf{F}(X)$ and Z as injective functions, the sequence of exchanges of subtrees of equal squash transforming $\mathsf{F}(X)$ into Z is equivalent to a permutation of $\{1, 2, \ldots, j + 2l\}$. We can express this permutation as a product of disjoint cycles. In terms of forests, each of these cycles permutes a set of subtrees of equal squash in the corresponding forest. Note that the set of trees permuted by such a cycle contains at most one leaf, namely the element of (12) completing the cycle, if needed.

Since these cycles act on disjoint sets of subtrees, we can assume the that sequence of subtree exchanges transforming $\mathsf{F}(X)$ into Z is a single cycle permuting subtrees of the same squash, at most one of which being a leaf. Suppose the cycle moves the subtree U of positive length to the position of the subtree V. If V has no parent, then it lies to the left of U since the parts of $\mathsf{F}(X)$ appear in nondecreasing order. If V has a parent, then again V lies to the left of U since otherwise the parent of V would have a larger node label than U. We conclude that the leftmost subtree permuted by the cycle is a leaf and the subtrees of positive length all move to the left, resulting in a forest which under E is lexicographically larger than X. ☐

Assembling the results above gives the main results of this section.

Proposition 20. $\mathsf{E} \circ \iota : kQ \to k\mathcal{M}^+$ *is surjective.*

Proof. Let X be a nondecreasing element of M^+ and put $P = P[\mathsf{F}(X)]$. Then $\iota(P) = \sum_{[U]\sim[\mathsf{F}(X)]}[U]$ by Lemma 18 so that taking $\mathcal{Y} = \mathsf{E}(\iota(P) - [\mathsf{F}(X)])$ we have $[X] < [Y]$ for all terms $[Y]$ of $\mathcal{Y}$ by Lemma 19. Then repeating the argument for all the terms of $\mathcal{Y}$ and subtracting the result from P gives an element of kQ mapping to $[X]$ under $\mathsf{E} \circ \iota$. ☐

Corollary 21. ι *is surjective modulo* $\ker \Delta$.

Proof. Let $X \in L$. We will show that some element of kQ maps under ι to an element of $k\mathcal{L}$ congruent to $[X]$ modulo $\ker \Delta$. If X has a node $\overset{i}{\widehat{Z_1 Z_2}}$ for which $Z_1 = Z_2 \in \mathbb{N}$ then $[X] \in \ker \Delta$ and we can take $P = 0$. Otherwise by Lemma 17 applied to all the terms of $[\mathsf{E}(X)]$ there exist $A \in M^+$ and $\mathcal{Y} \in \mathcal{N} + \mathcal{J}$ such that $[\mathsf{E}(X)] = [A] + \mathcal{Y}$. Applying F we have $[X] \cong [\mathsf{F}(A)] \pmod{\ker \Delta}$. By Proposition 20 we have $P \in kQ$ such that $\mathsf{E}(\iota(P)) = [A]$ so that $\iota(P) - [\mathsf{F}(A)] \in \ker \mathsf{E} \subseteq \ker \Delta$. ☐

Proposition 22. Q_n *is the ordinary quiver of* $\Sigma(\mathfrak{S}_n)$.

Proof. Let $I = \iota^{-1}(\ker \Delta)$ so that $kQ_n/I \cong \iota(kQ_n)/\ker \Delta$ since ι is injective by Corollary 12. But $\iota(kQ_n)/\ker \Delta = k\mathcal{L}_n/\ker \Delta$ by Corollary 21 and $k\mathcal{L}_n/\ker \Delta \cong \Sigma(\mathfrak{S}_n)$ by Theorem 7. Let R be the Jacobson radical of kQ_n. Then R is generated by all paths of Q_n of positive length. Since Q_n is the ordinary quiver of any quotient of kQ_n by an ideal contained in R^2 by [1, Lemma 3.6] it suffices to show that $I \subseteq R^2$.

Let P be any element of I. By multiplying P on the left and on the right by various vertices of Q_n we can split P into a sum of elements of I all of whose terms have the same source and destination. We therefore assume that all the terms of P have the same source and destination and hence the same length. If this length were zero or one, then P would be a multiple of a vertex or an edge. But $\Delta(\iota(p)) = [p] \neq 0$ for all vertices p, while $\Delta(\iota(e)) \neq 0$ for all edges e by Lemma 9. Therefore $P \in R^2$. $\square$

12. The Relations

In this section we state our conjecture on the relations for the quiver presentation of $\Sigma(\mathfrak{S}_n)$. Let $\mathcal{R} \subseteq k\mathcal{B}^*$ be the set of elements

$$\left\langle\begin{matrix} a & c \\ b & d \end{matrix}\right| - \left\langle\begin{matrix} c & a \\ d & b \end{matrix}\right| \qquad \text{where} \qquad a+b \notin \{c, d\} \qquad \text{and} \qquad c+d \notin \{a, b\} \tag{13}$$

and the elements

$$\left\langle\begin{matrix} a & c & x \\ b & d & y \end{matrix}\right| + \left\langle\begin{matrix} x & a & c \\ y & b & d \end{matrix}\right| - \left\langle\begin{matrix} a & x & c \\ b & y & d \end{matrix}\right| - \left\langle\begin{matrix} c & x & a \\ d & y & b \end{matrix}\right| \tag{14}$$

where a, b, c, d satisfy the condition in (13) and either

(1) $a + b = c + d \in \{x, y\}$ or
(2) $x + y \in \{a, b\} \cap \{c, d\}$.

The elements of $\mathcal{R}$ are called *branch relations*. The following proposition shows that the branch relations produce relations by applying them to vertices of Q.

Proposition 23. *If* $R \in \mathcal{R}$ *then* $p.R \in \ker(E \circ \iota)$ *for all partitions* p.

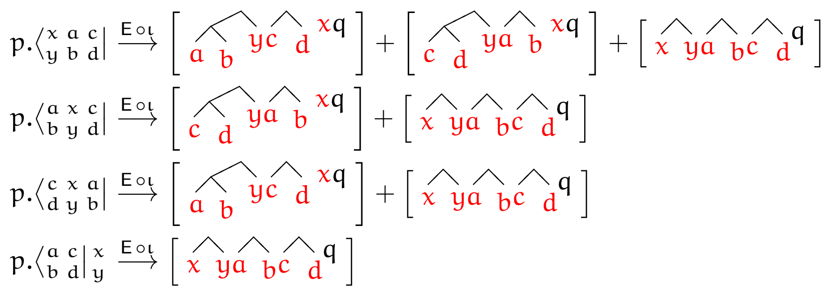
Proof. Suppose $R = \left\langle\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right| - \left\langle\begin{smallmatrix} c & a \\ d & b \end{smallmatrix}\right|$ where $a, b, c, d \in \mathbb{N}$ satisfy the condition in (13). Then $\iota\left(p.\left\langle\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right|\right) = \left[\; a \overset{1}{\wedge} b \; c \overset{2}{\wedge} d \quad q \;\right]$ and $\iota\left(p.\left\langle\begin{smallmatrix} c & a \\ d & b \end{smallmatrix}\right|\right) = \left[\; a \overset{2}{\wedge} b \; c \overset{1}{\wedge} d \quad q \;\right]$ whenever $p = (a+b)(c+d)\, q$ for some partition q, while both expressions are zero otherwise. Thus $E(\iota(p.R)) = 0$ for all partitions p.

Now let R be the element in (14) and suppose $a, b, c, d, x, y \in \mathbb{N}$ satisfy condition (1) of the definition of $\mathcal{R}$. Specifically, we assume that $a + b = c + d = x$, but the argument can be modified if $a + b = c + d = y$. In each of the cases that

(1) p has at least one part $x + y$ and exactly one part x
(2) p has at least one part $x + y$ and two or more parts x
(3) p has no part $x + y$ or no part x

the image of $p.R$ can be calculated explicitly. In the third situation $E \circ \iota$ maps all four terms of $p.R$ to zero. In the second situation we take p to be the partition

$(x + y)\, xxq$ where q is any partition. Then we calculate



so that $\mathsf{E}(\iota(p.R)) = 0$. The first situation is similar to the second and the calculation in the case that a, b, c, d, x, y satisfy condition (2) of the definition of $\mathcal{R}$ is similar to the calculation above. $\square$

For unlabeled trees X, Y, Z we denote $\widehat{X\,\widehat{Y\,Z}} + \widehat{Z\,\widehat{X\,Y}} + \widehat{Y\,\widehat{Z\,X}}$ by $\mathsf{j}(X, Y, Z)$. Suppose that $A = \sum_{i=1}^{m} A_i$ is an aligned rendering of $\mathsf{j}(X, Y, Z)$ where $A_1, A_2, \ldots, A_m$ are aligned unlabeled forests. We observe that if it exists, A may have more or fewer than three terms and satisfies $A - \mathsf{j}(X, Y, Z) \in \ker \pi$ by Lemma 17. But since $\mathsf{j}(X, Y, Z) \in \ker \pi$ we have $A \in \ker \pi$. Inserting any partition q into the terms of A and taking Pólya classes, we have an element $P \in kQ$ such that $\mathsf{E}(\iota(P)) = \sum_{i=1}^{m} [Aq]$ by Proposition 20. Then $P \in \ker(\Delta \circ \iota)$ so that P is a relation.

Let $\mathcal{S}$ be a set of elements $P \in kQ$ for which $\mathsf{E}(\iota(P)) = \sum_{i=1}^{m} [A_i q]$ where q is any composition and $\sum_{i=1}^{m} A_i$ is an aligned rendering of an element of the form

(1) $\mathsf{j}(x, y, z)$ where $x < y < z$ are natural numbers such that $x + y \neq z$
(2) $\mathsf{j}\left(\widehat{x_1\, x_2}, y, z\right)$ where $x_1 < x_2$ and $y < z$ are natural numbers such that $x_1 + x_2 \in \{y, z, y + z\}$.

Then the elements of $\mathcal{S}$ are relations for the quiver presentation of $\Sigma(\mathfrak{S}_n)$. Observe that elements of the form $\mathsf{j}(x, y, z)$ with $x, y, z \in \mathbb{N}$ have only one possible aligned rendering, while elements of form $\mathsf{j}\left(\widehat{x_1\, x_2}, y, z\right)$ with $x_1, x_2, y, z \in \mathbb{N}$ have only one "useful" aligned rendering. For example, the term $\widehat{6\,\widehat{\widehat{1\,2}\,3}}$ of $\mathsf{j}\left(\widehat{1\,2}, 3, 6\right)$ has the two aligned renderings shown in (9) but only the second can be used to construct a relation, since the terms of the first aligned rendering cancel the other terms of $\mathsf{j}\left(\widehat{1\,2}, 3, 6\right)$.

We conjecture that the relations above generate the ideal of relations for the quiver presentation of $\Sigma(\mathfrak{S}_n)$ in the following way.

Conjecture 24. *The descent algebra* $\Sigma(\mathfrak{S}_n)$ *has a presentation as the path algebra* kQ_n *subject to the relations* $\mathcal{S} \cap kQ_n$ *and* $p.R$ *for all partitions* p *of* n *and all* $R \in \mathcal{R}$. *In particular, the relations all have length two or three.*

We have verified Conjecture 24 through computer calculation for $n \leqslant 15$. In fact, we have implemented a procedure in GAP [11] which calculates minimal projective resolutions over the algebra $A = kQ_n / \ker(\Delta \circ \iota)$ of the simple module $(A / \operatorname{Rad} A)\, p$ for all partitions p of n. One result of the calculation is a minimal generating set

FIGURE 2. Numbers of Relations

n	Branch	Jacobi	Minimal
6	0	1	1
7	1	3	4
8	4	7	11
9	10	14	24
10	22	29	48
11	44	51	90
12	86	89	160
13	152	146	270
14	265	240	444
15	441	369	705

of $\mathsf{ker}\,(\Delta \circ \iota)$ which can be used to confirm that the presentation in Conjecture 24 is correct for small n.

Figure 2 shows the minimal number of relations for the presentation of $\Sigma\,(\mathfrak{S}_n)$ for $n \leqslant 15$. The table also shows the numbers of branch and Jacobi relations. Note that when $n = 10$ the number of branch and Jacobi relations exceeds the size of the minimal generating set due to the algebraic dependence of the branch relations and the Jacobi relations. Nonetheless, the ideal generated by the branch and Jacobi relations is exactly $\mathsf{ker}\,(\Delta \circ \iota)$ in every case shown in Figure 2.

13. EXAMPLE

As an example of Conjecture 24 we calculate the presentation for $\Sigma\,(\mathfrak{S}_8)$. The quiver for this presentation is shown in Figure 1. To calculate the branch relations, we list all $R \in \mathcal{R}$ and apply them to all vertices p of Q_8. Those resulting in nonzero relations are shown in the column labeled ¶ of Table 1. To calculate the Jacobi relations, we list all tuples x, y, z and x_1, x_2, y, z satisfying the conditions in the definition of $\mathcal{S}$. For each partition q for which $p = xyzq$ or $p = x_1x_2yzq$ is a composition of n we find elements $P \in kQ$ for which $E\,(\iota\,(P)) = \sum_{i=1}^{m} [A_i q]$ where $\sum_{i=1}^{m} A_i$ is an aligned rendering of $j\,(x, y, z)$ or $j\left(\widehat{x_1\, x_2}\,, y, z\right)$. These relations are also shown in Table 1.

As mentioned in §12 we have verified through a computer calculation that the quotient of kQ_8 by the ideal generated by the elements in Table 1 is isomorphic to $\Sigma\,(\mathfrak{S}_8)$.

REFERENCES

[1] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*, volume 65 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006. Techniques of representation theory.

[2] M. D. Atkinson. A new proof of a theorem of Solomon. *Bull. London Math. Soc.*, 18(4):351–354, 1986.

[3] M. D. Atkinson. Solomon's descent algebra revisited. *Bull. London Math. Soc.*, 24(6):545–551, 1992.

TABLE 1. Relations for $\Sigma(\mathfrak{S}_8)$

p	R	P
35	$\left\langle\begin{smallmatrix}1&1\\2&4\end{smallmatrix}\right\| - \left\langle\begin{smallmatrix}1&1\\4&2\end{smallmatrix}\right\|$	$jq - kt$
134	$\left\langle\begin{smallmatrix}1&1&1\\2&2&3\end{smallmatrix}\right\| + \left\langle\begin{smallmatrix}1&1&1\\3&2&2\end{smallmatrix}\right\| - 2\left\langle\begin{smallmatrix}1&1&1\\2&3&2\end{smallmatrix}\right\|$	$adm - 2ack$
35	$\left\langle\begin{smallmatrix}1&1&2\\2&2&3\end{smallmatrix}\right\| + \left\langle\begin{smallmatrix}2&1&1\\3&2&2\end{smallmatrix}\right\| - 2\left\langle\begin{smallmatrix}1&2&1\\2&3&2\end{smallmatrix}\right\|$	$bis - 2bhq$
44	$\left\langle\begin{smallmatrix}1&1&1\\3&3&2\end{smallmatrix}\right\| + \left\langle\begin{smallmatrix}1&1&1\\2&3&3\end{smallmatrix}\right\| - 2\left\langle\begin{smallmatrix}1&1&1\\3&2&3\end{smallmatrix}\right\|$	$dmu - 2cku$
116	$j\left(\widehat{1\ \ 2}, 1, 2\right)$	$acl - aen$
17	$j(1, 2, 4)$	$jr + kv - lw$
17	$j\left(\widehat{1\ \ 2}, 1, 3\right)$	$2ckv + clw - dmv - enw$
26	$j\left(\widehat{1\ \ 2}, 1, 2\right)$	$bgo - bhp$
8	$j(1, 2, 5)$	$px + qy - rz$
8	$j\left(\widehat{1\ \ 3}, 1, 3\right)$	$mty - mvz$
8	$j\left(\widehat{1\ \ 2}, 2, 3\right)$	$gox - hpx - 2hqy - isy$

[4] F. Bergeron and N. Bergeron. A decomposition of the descent algebra of the hyperoctahedral group. I. *J. Algebra*, 148(1):86–97, 1992.

[5] F. Bergeron, N. Bergeron, R. B. Howlett, and D. E. Taylor. A decomposition of the descent algebra of a finite Coxeter group. *J. Algebraic Combin.*, 1(1):23–44, 1992.

[6] François Bergeron and Nantel Bergeron. Orthogonal idempotents in the descent algebra of B_n and applications. *J. Pure Appl. Algebra*, 79(2):109–129, 1992.

[7] Dieter Blessenohl and Hartmut Laue. On the descending Loewy series of Solomon's descent algebra. *J. Algebra*, 180(3):698–724, 1996.

[8] Dieter Blessenohl and Hartmut Laue. The module structure of Solomon's descent algebra. *J. Aust. Math. Soc.*, 72(3):317–333, 2002.

[9] Dieter Blessenohl and Manfred Schocker. *Noncommutative character theory of the symmetric group*. Imperial College Press, London, 2005.

[10] C. Bonnafé and G. Pfeiffer. Around Solomon's descent algebras. *Algebr. Represent. Theory*, 11(6):577–602, 2008.

[11] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.4.12*, 2008.

[12] A. M. Garsia and C. Reutenauer. A decomposition of Solomon's descent algebra. *Adv. Math.*, 77(2):189–262, 1989.

[13] Götz Pfeiffer. Quiver presentations for descent algebras of exceptional type. 2008.

[14] Götz Pfeiffer. A quiver presentation for Solomon's descent algebra. *Adv. Math.*, 220(5):1428–1465, 2009.

[15] Franco V. Saliola. On the quiver of the descent algebra. *J. Algebra*, 320(11):3866–3894, 2008.

[16] Franco V. Saliola. The Loewy length of the descent algebra of type D. *Algebr. Represent. Theory*, 13(2):243–254, 2010.

[17] Manfred Schocker. The descent algebra of the symmetric group. In *Representations of finite dimensional algebras and related topics in Lie theory and geometry*, volume 40 of *Fields Inst. Commun.*, pages 145–161. Amer. Math. Soc., Providence, RI, 2004.

[18] Louis Solomon. A Mackey formula in the group ring of a Coxeter group. *J. Algebra*, 41(2):255–264, 1976.

Ruhr-Universität Bochum, Fakultät für Mathematik
E-mail address: `marcus.bishop@rub.de`

School of Mathematics, Statistics, and Applied Mathematics, National University of Ireland, Galway
E-mail address: `Goetz.Pfeiffer@nuigalway.ie`