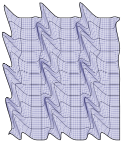

Counter-Intuitive Acousto-Elasticity

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Co-Authors:
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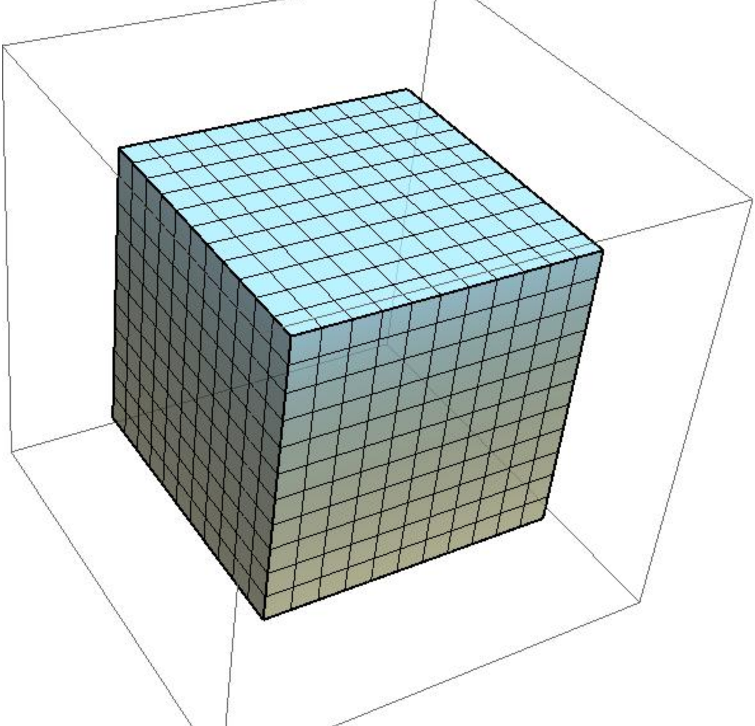
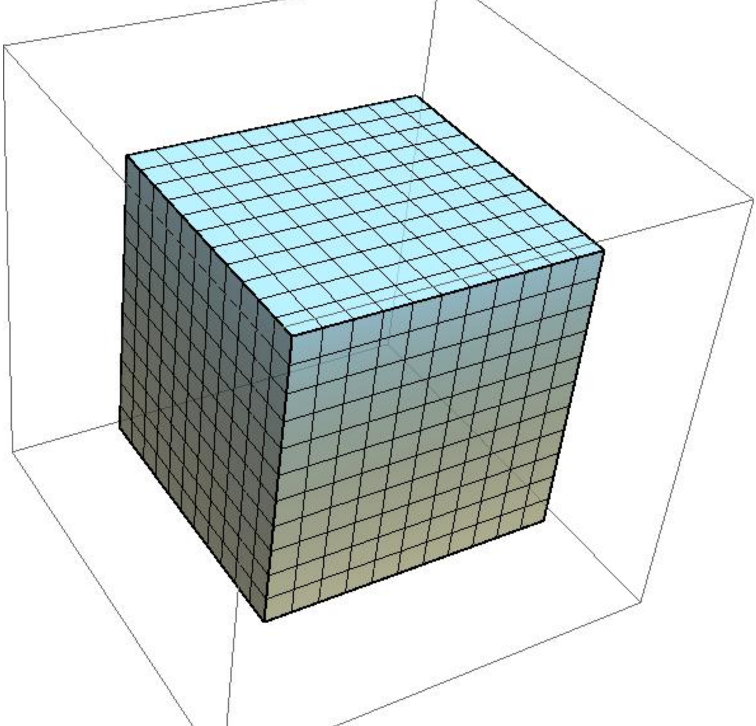
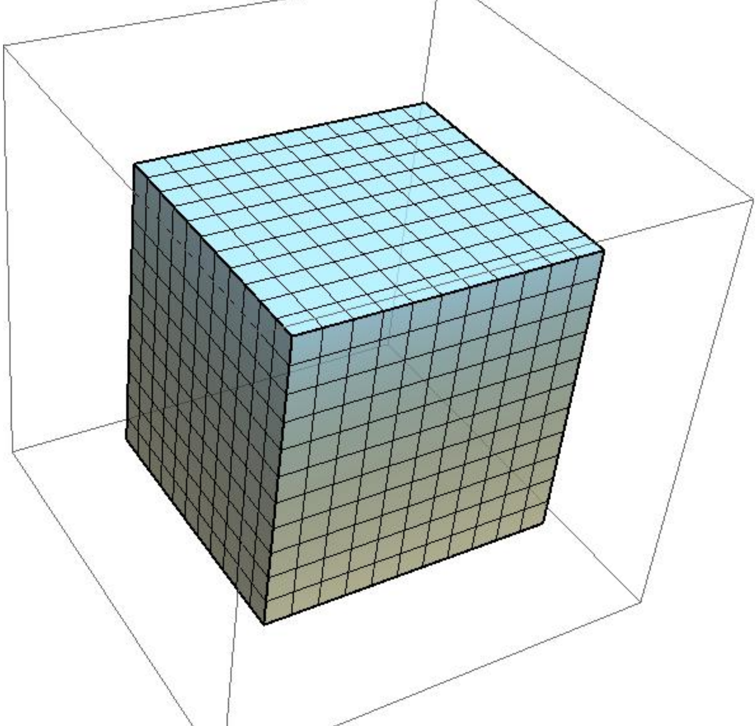
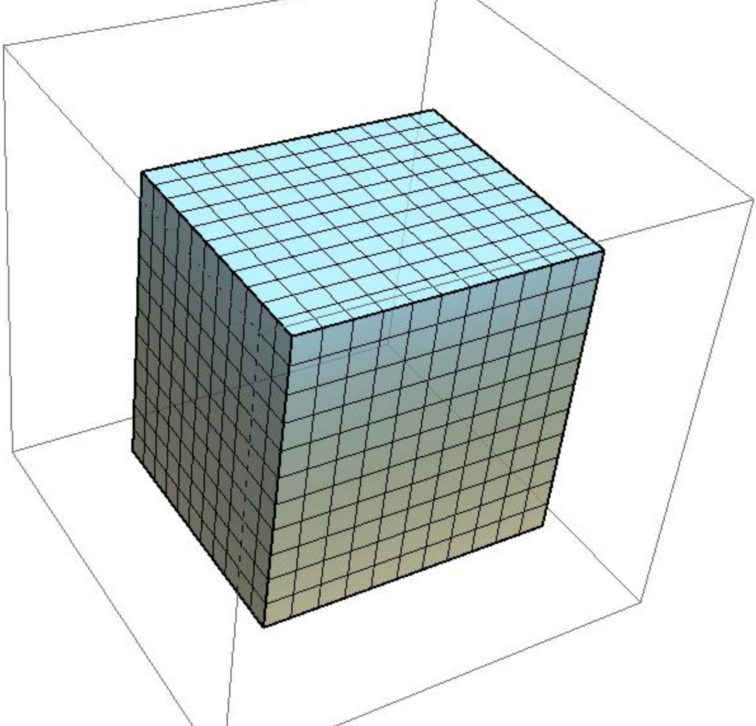
National University of Ireland Galway

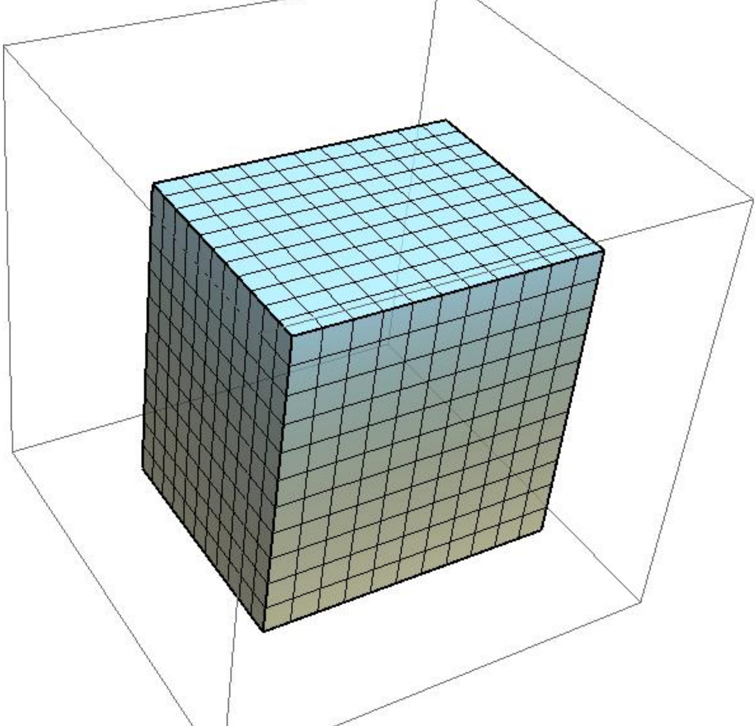
NUI Galway
OÉ Gaillimh

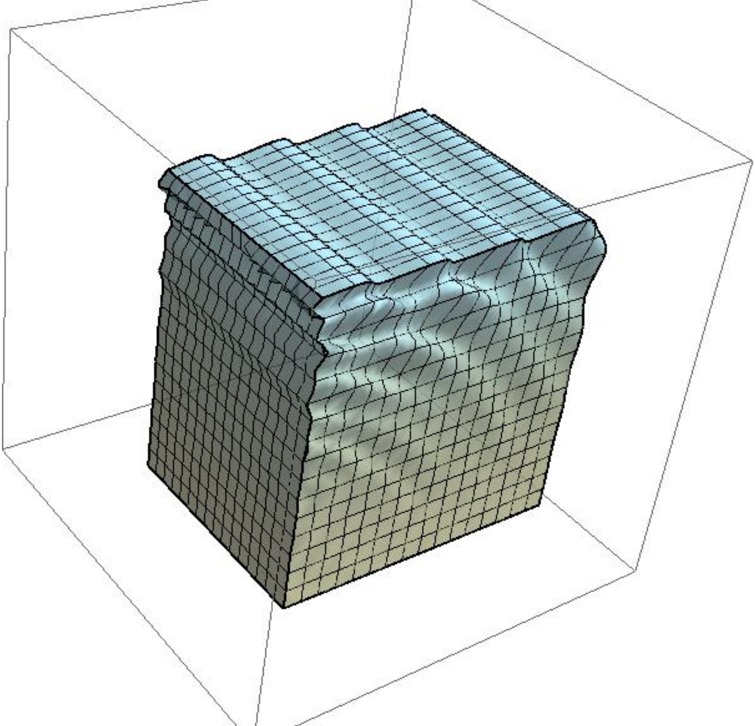


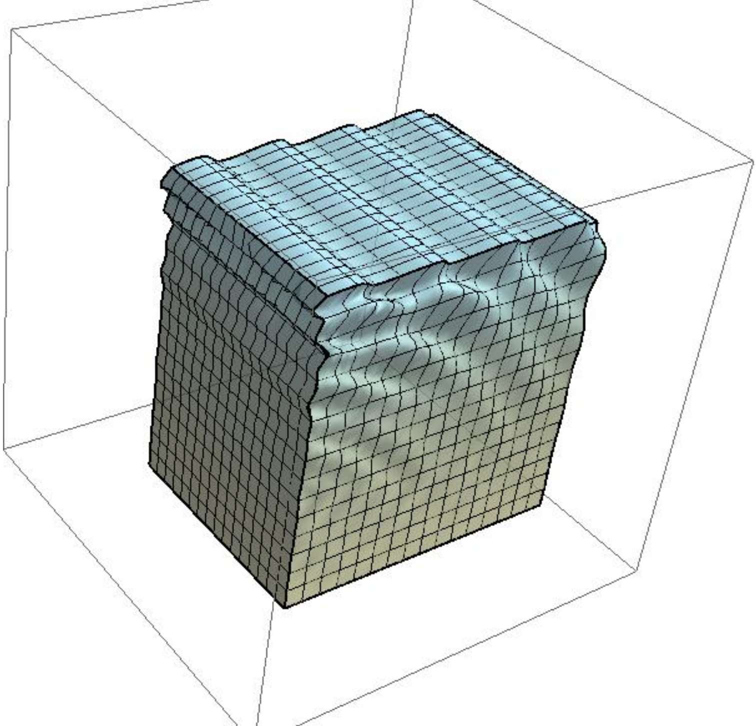


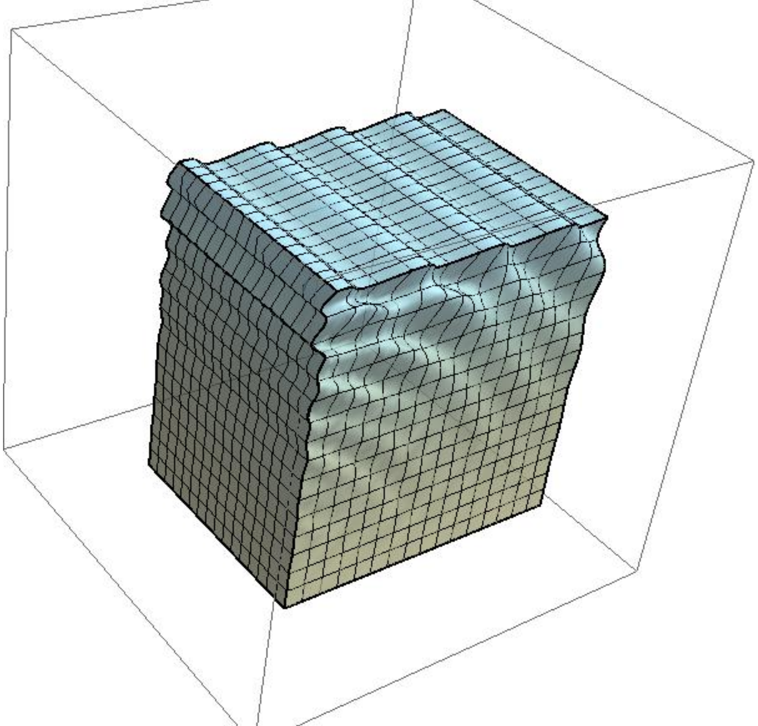


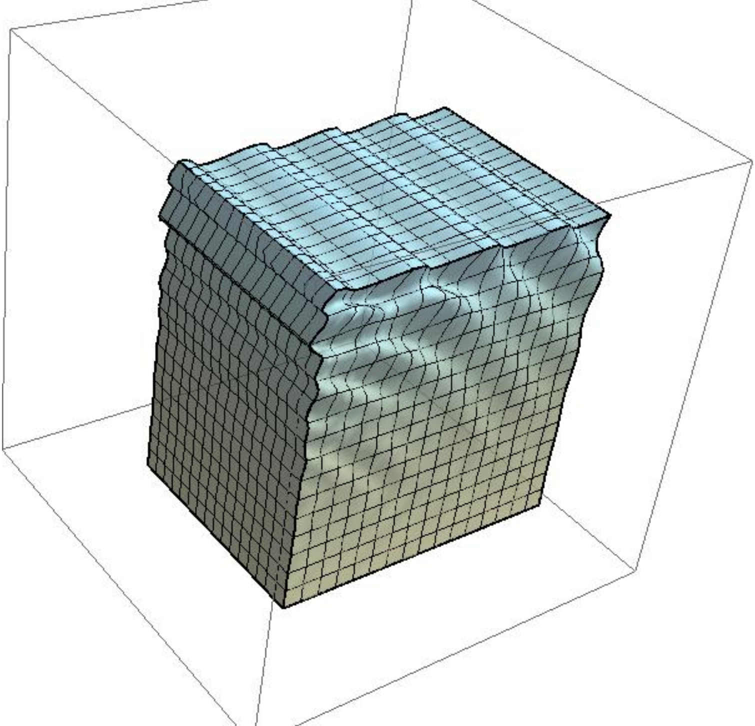


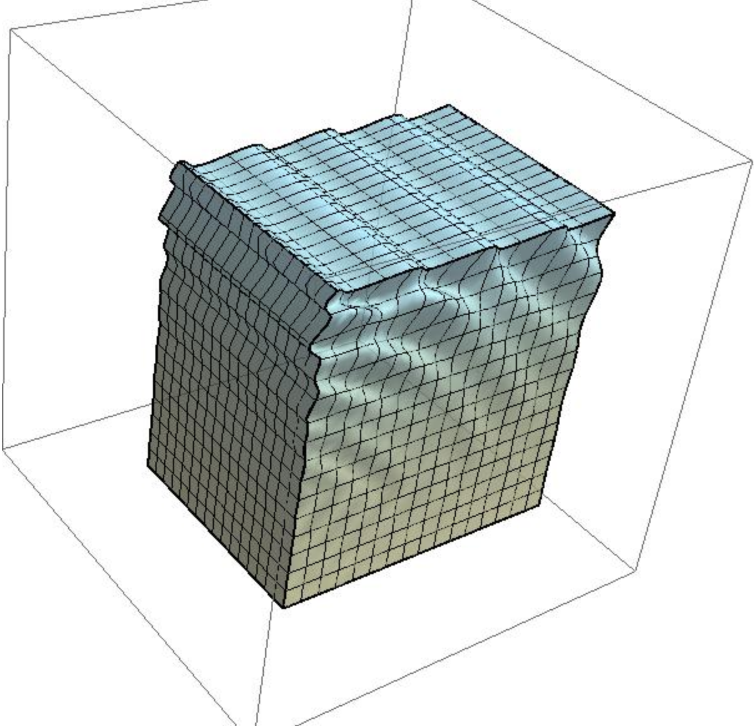


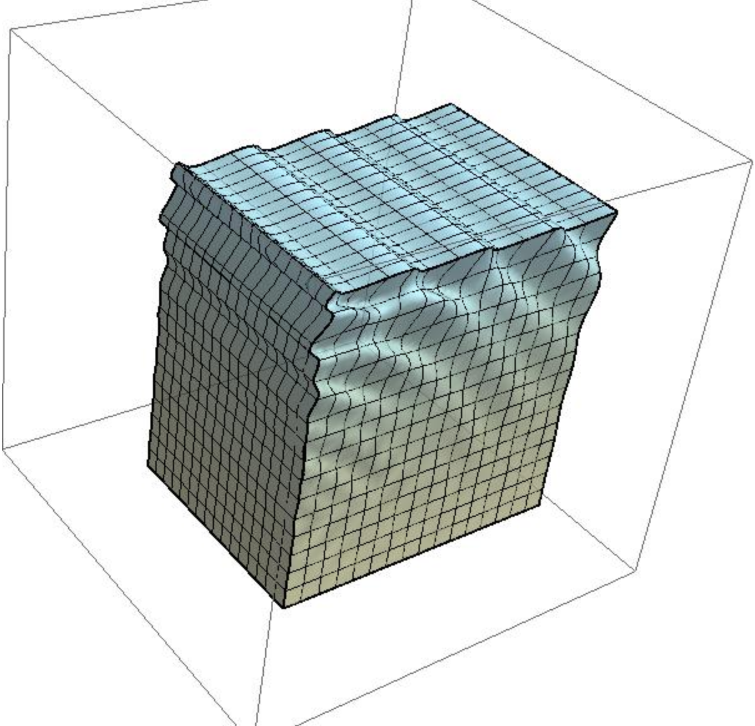


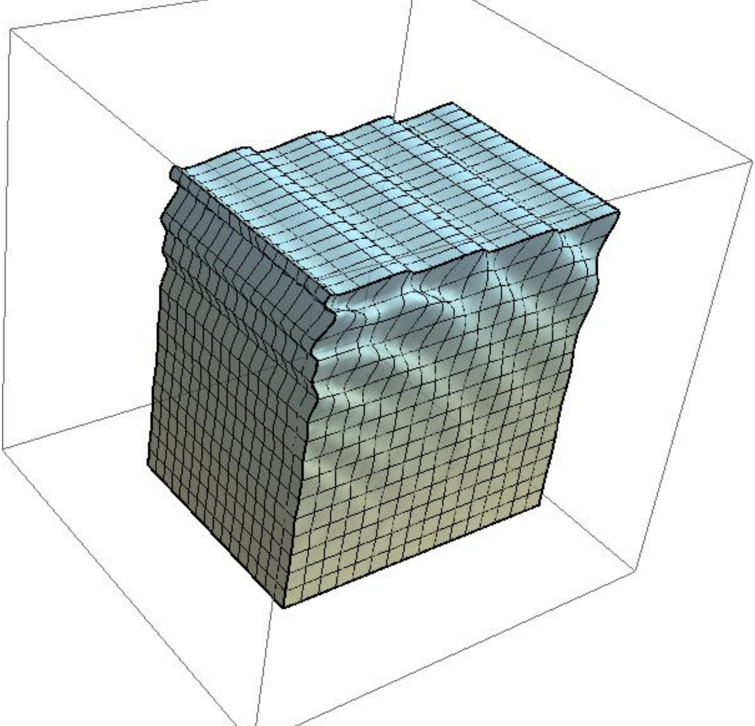


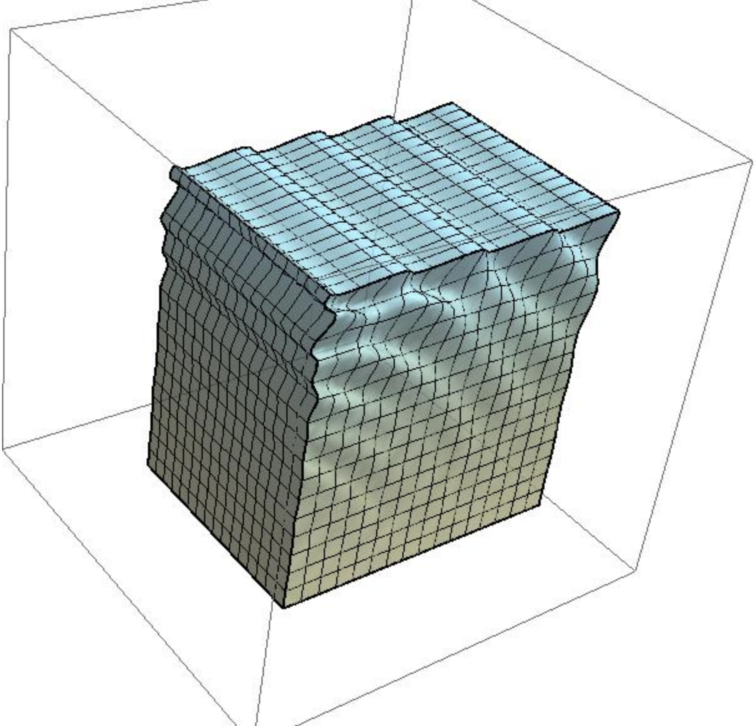


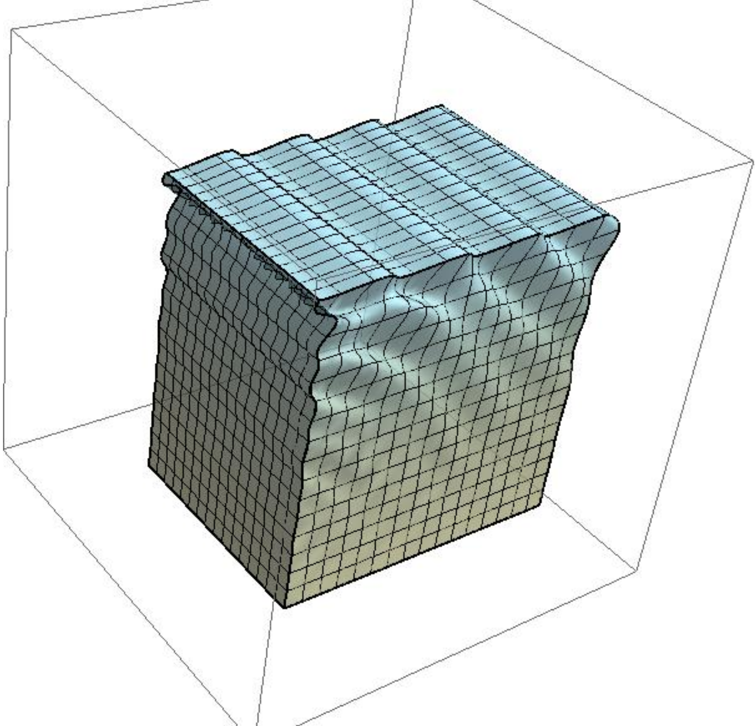






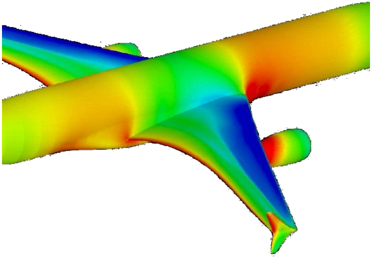






Waves tell us about stress

Measure wave velocity to uncover stress field ⟶



⟵ Predict wave velocity from a known stress field

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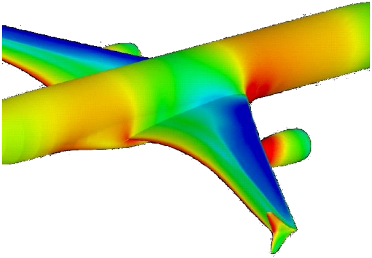
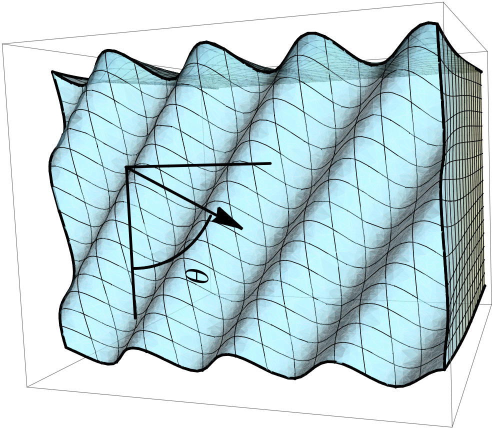


⟵ Predict wave velocity from a known stress field

Because rocks behave approximately like a big rubber ball.

Wavefront Angle from direction of Greatest Compression



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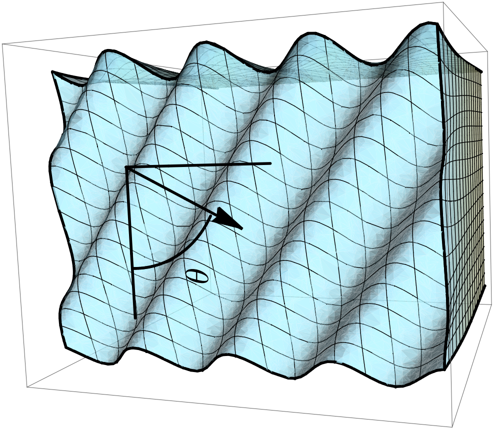

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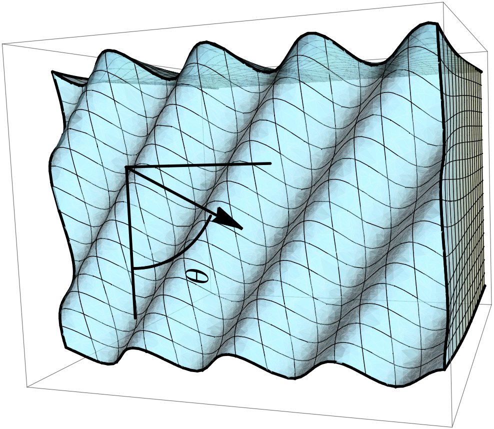
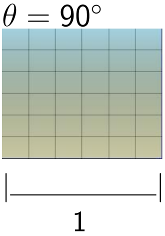


$$\mathbf{u}(x,y,z) = \mathcal{U}(y)e^{ik(x\cos\theta + z\sin\theta - vt)}$$ (Incremental displacement)

$$\lim_{y\to\infty} \mathcal{U}(y) = 0 \text{ and } \mathcal{V}(0) = \mathbf{0}$$ (Boundary conditions)

(CORRECT DECAY & ZERO SURFACE TRACTION)

What is intuitive about a deformed isotropic material?



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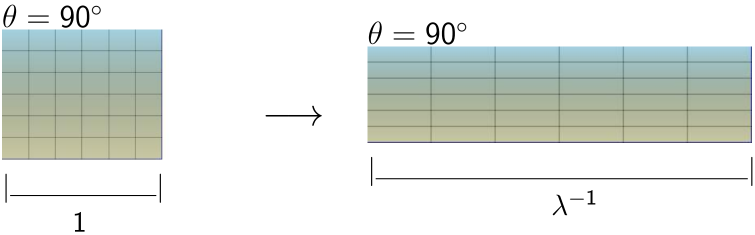

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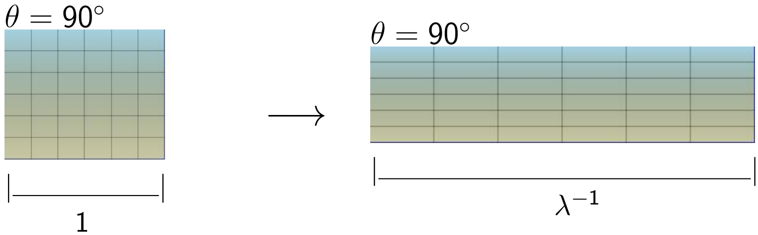

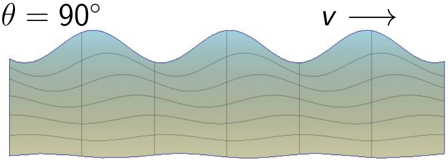

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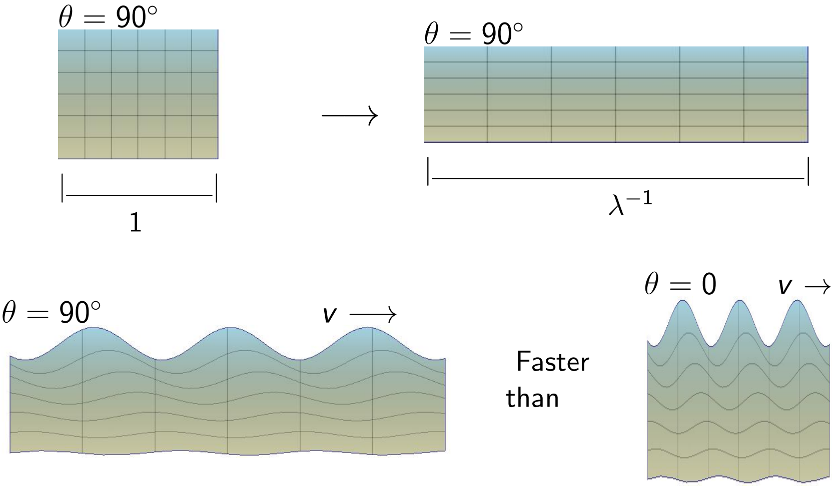

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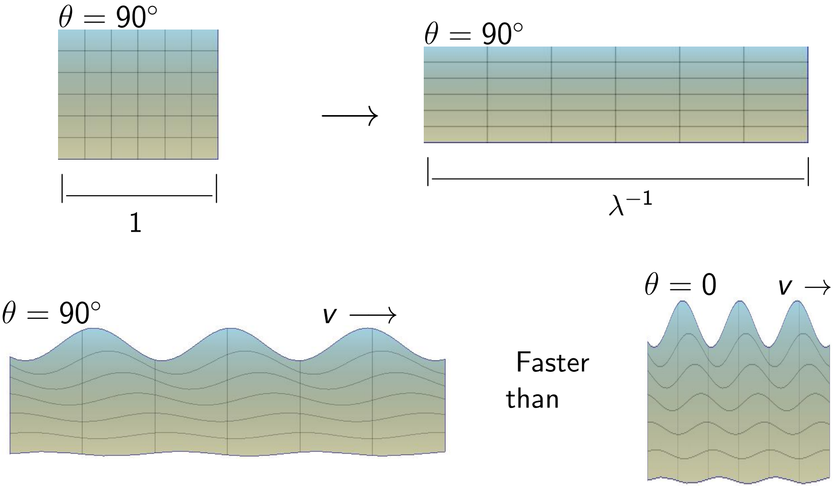

ISOTROPIC:

Direction of greatest stress = Direction of greatest strain

Inuitive Infinitesimal Prestress

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$$v_R(0) = v_R^0 + C_1(\sigma_1 + \sigma_3) - C_2(\sigma_1 - \sigma_3) \quad \leftarrow \textbf{Min Velocity}$$
$$v_R(\pi/2) = v_R^0 + C_1(\sigma_1 + \sigma_3) + C_2(\sigma_1 - \sigma_3) \quad \leftarrow \textbf{Max Velocity}$$

If the principal pre-stresses along the surface σ_1 and σ_3 satisfy $\sigma_1 > \sigma_2$. Where C_1 and C_2 are complicated constants.

Nonlinear Elastic Results

$$W = \frac{\lambda_0}{2} i_1^2 + \mu_0 i_2 + \frac{A}{3} i_3 + B i_1 i_2 + \frac{C}{3} i_1^3 \quad \text{(Landau coefficients)}$$

For nonlinear elasticity, all bets are off...

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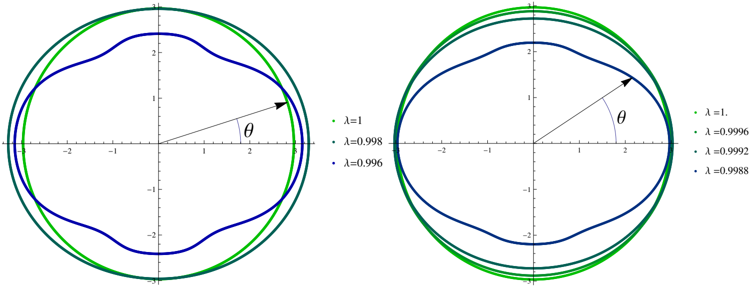

Figure: Speed profiles for surface waves (plotted as $v\sqrt{\rho}$) in Concrete subject to Uniaxial stress (left) and to Shear Stress (right).

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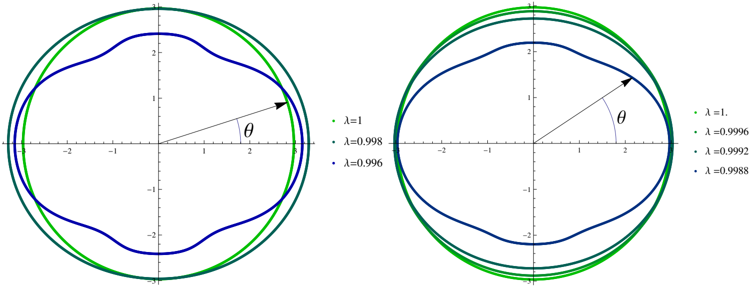
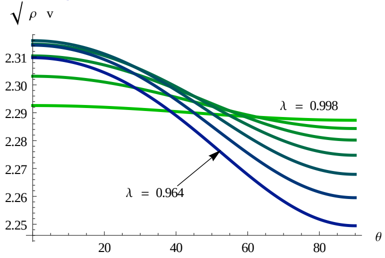


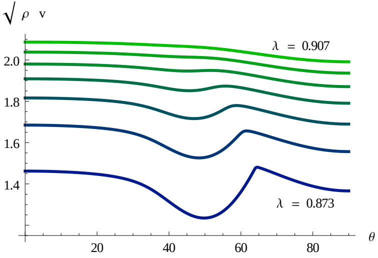
Figure: Speed profiles for surface waves (plotted as $v\sqrt{\rho}$) in Concrete subject to Uniaxial stress (left) and to Shear Stress (right).

- The sinusoidal regularity was lost early, for strains less than 1% (though the stress is reasonable.)

Nonlinear Elastic Results

Nickel Uniaxial





Nonlinear Elastic Results

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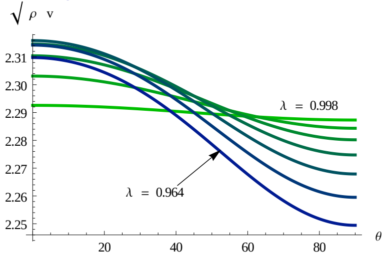
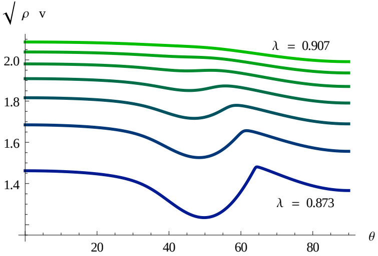
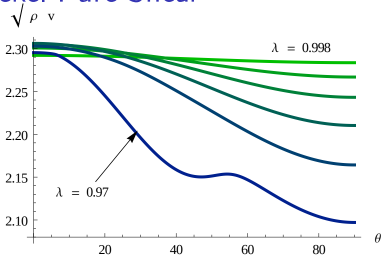
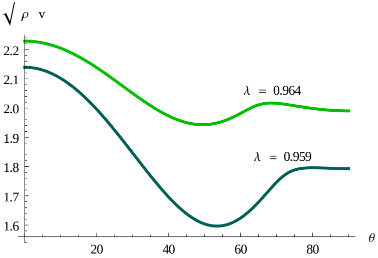




Nickel Pure Shear





Nonlinear Elastic Results

The higher the third-order constants (Landau/Murnaghan) the earlier the onset of nonlinear effects.

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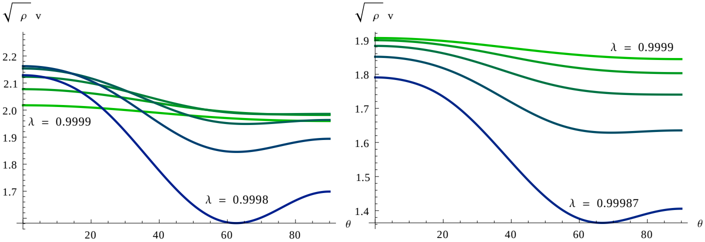
Uniaxial and Pure Shear Berea



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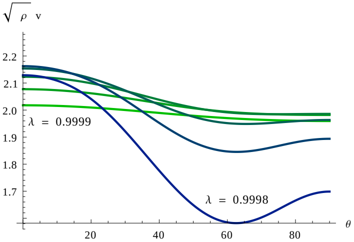
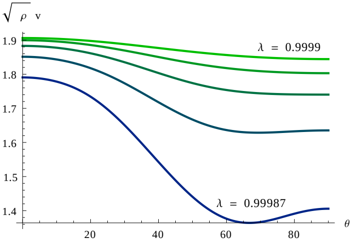
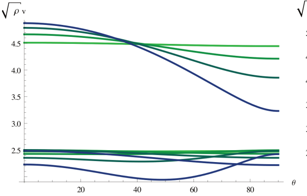
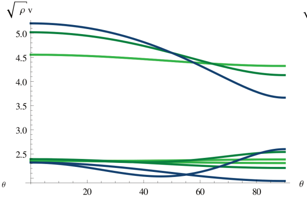
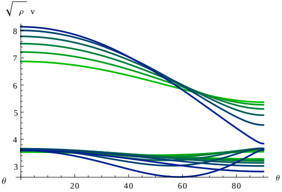




Bulk Waves (Nickel, Steel and Concrete)







Matrix Impedance Method

Incremental quantities:

$$\mathbf{u}(x,y,t) = \underbrace{\mathcal{U}(y)\mathrm{e}^{\mathrm{i}kx-\mathrm{i}vt}}_{Displacement} \qquad \mathbf{v}(x,y,t) = \underbrace{-\mathrm{i}\mathcal{V}(y)\mathrm{e}^{\mathrm{i}kx-\mathrm{i}vt}}_{Normal Traction}$$

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We've identified the object of study $Z(v)$, now for some *magic*.

Matrix Impedance Magic

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mathrm{div}\ \boldsymbol{\sigma} \implies -v^2 \rho \mathbf{u}^* \cdot \mathbf{u} = \mathbf{u}^* \cdot \mathrm{div}\ \boldsymbol{\sigma},$$

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$$\mathcal{U}^*(0) \cdot \frac{\partial Z(v)}{\partial v}\mathcal{U}(0) = -2v \int_0^\infty \rho \mathcal{U}^*(y) \cdot \mathcal{U}(y)dy,$$

positive definite $Z(0)$ with monotone decreasing Eigenvalues!

Analytic Solution

from balance of momentum we get an algebraic Riccati equation,

$$\boxed{H^{\dagger}(v)H(v) = Q - \rho v^2 I \quad \text{and} \quad Z(v) = T^{1/2}H(v) - \mathrm{i}R,}$$

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The restriction

$$\boxed{Z(v) > 0}$$

uniquely defines $Z(v)$, which is then easy to find numerically for each v.

More results

This procedure works for any elastic strain-energy function, for example...

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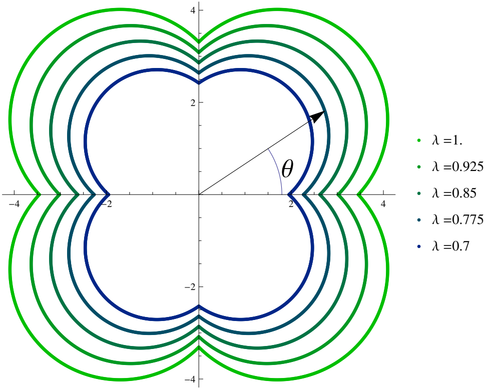
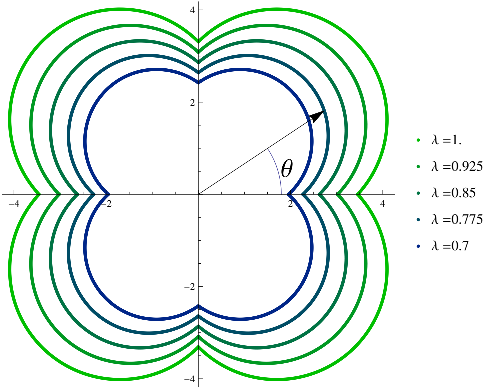


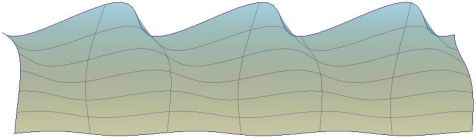
More results

This procedure works for any elastic strain-energy function, for example...

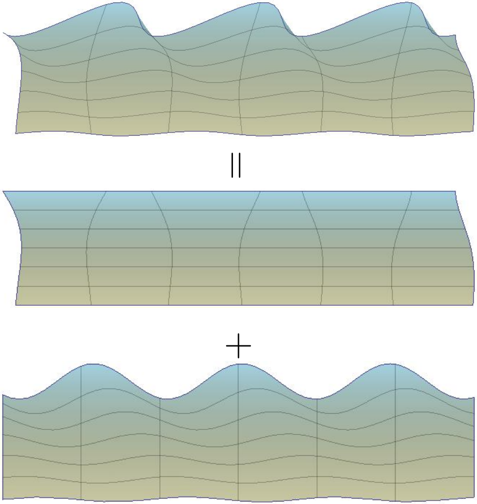


A model for skin, that has a neo-hookean matrix with fibers. This is an example of shear against the skin fibers.

What happened to our intuition?

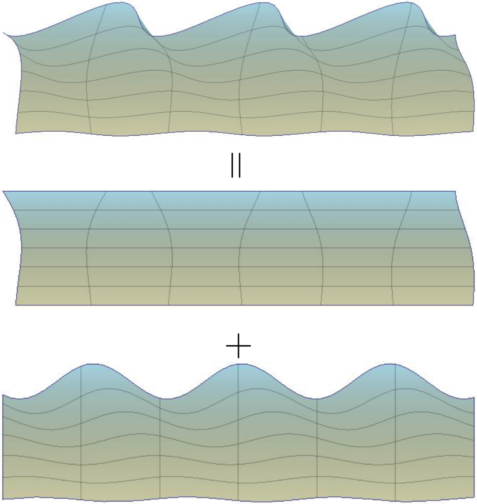


What happened to our intuition?



Any questions?

What happened to our intuition?



Any questions?

Thanks for listening and hope you enjoyed the talk!

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