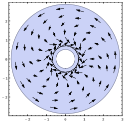

The eye of sauron

The Potential Energy of Residually Stressed Solids

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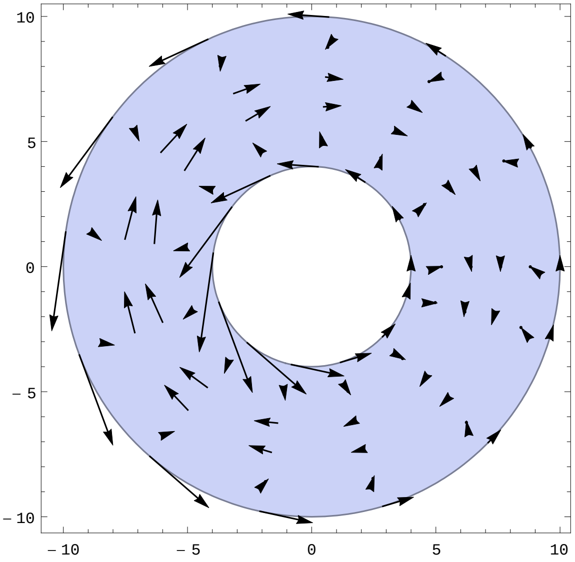
Measurement from hell



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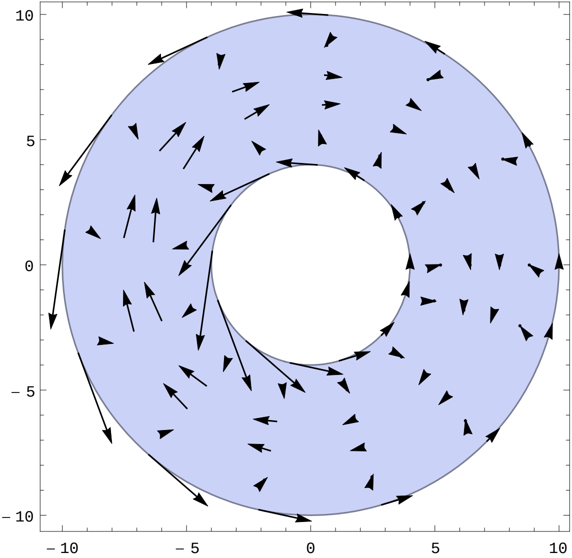
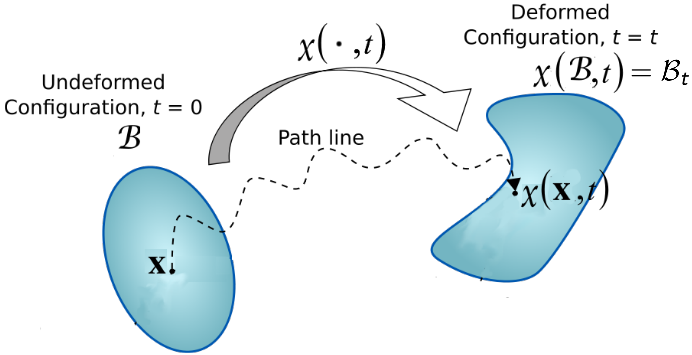


How do we deal with this? Energy $W(\mathbf{B}, \boldsymbol{\tau})$

Continuum Blob Mechanics

We apply the laws of physics to a continuous blob which is moving through space.



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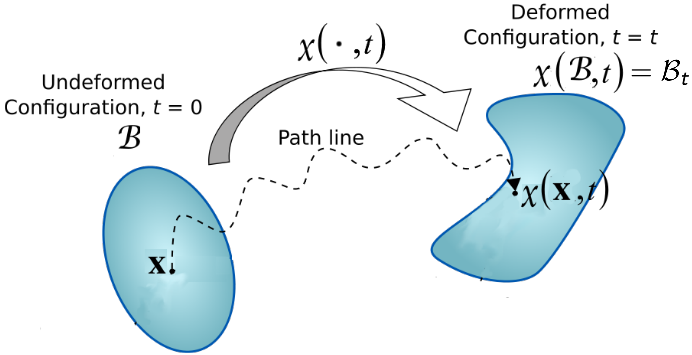
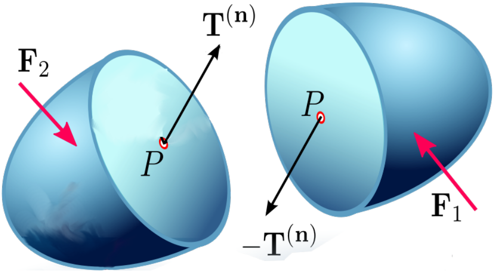


With each position $x = \chi(X, t)$ we associate denisties such as mass $\rho(x, t)$ and stress $\mathbf{T}^{(\mathbf{n})}(x, t)$.

Continuum Blob Mechanics

To describe the forces $\mathbf{F}_1(\mathbf{X}, t)$, $\mathbf{F}_2(\mathbf{X}, t)$, $\mathbf{T}^{(\mathbf{n})}(\mathbf{X}, t)$, $\mathbf{n} \in T_x\mathcal{B}_t$, we make an imaginary slice



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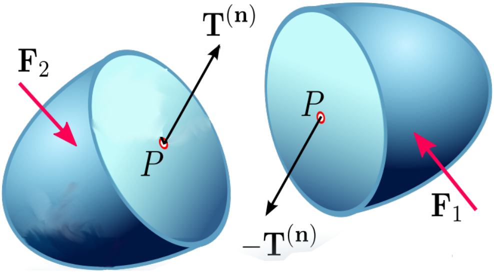

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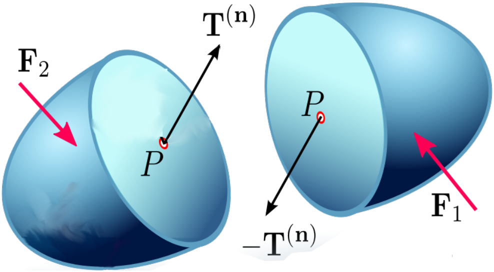


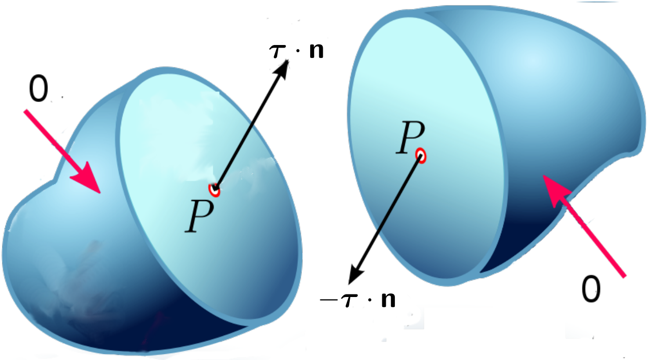
The forces $\mathbf{F}_1$ and $\mathbf{F}_2$ may be determined by Boundary conditions. While the internal stress $\mathbf{T}^{(\mathbf{n})}$ can be written as $\mathbf{T}^{(\mathbf{n})} = \boldsymbol{\sigma} \cdot \mathbf{n}$.

(One of Cauchy's many theories).

- We call $\boldsymbol{\sigma}$ a stress tensor, with $\boldsymbol{\sigma}(X, t) \in T_x\mathcal{B}_t \otimes T_x\mathcal{B}_t$.

Continuum Blob Mechanics

The residual stress tensor $\boldsymbol{\tau} = \boldsymbol{\sigma}$, when all external load is removed.



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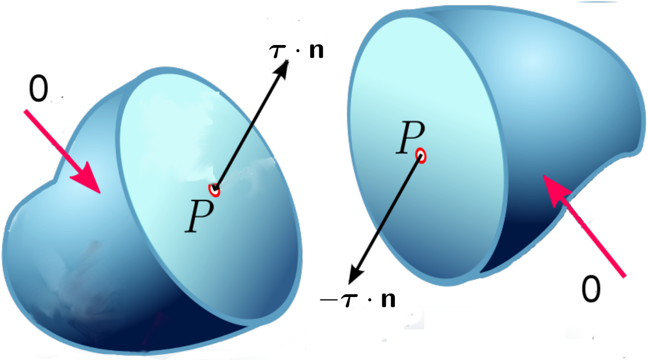
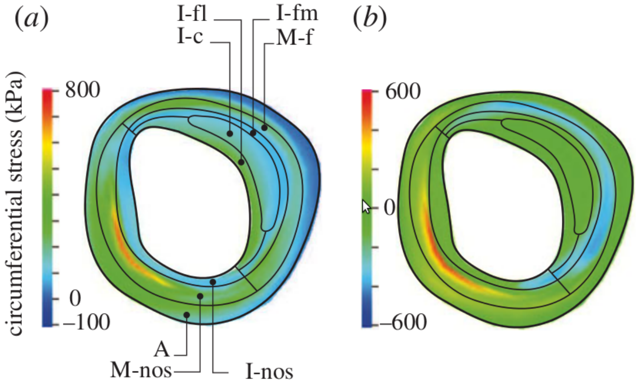


To incorporate the residual stress $\boldsymbol{\tau}$ into the mechanics, we use the above as a reference state.

Continuum Blob Mechanics

The circumferential stress in the cross section of an artery:



(a) is unloaded, (b) is loaded (assuming isotropy).

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The usual laws of physics alone are not enough to determine the stress $\boldsymbol{\sigma}$ in terms of χ and $\boldsymbol{\tau}$.

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The result $W(X, t) = \hat{W}(\mathbf{C}(X, t), \boldsymbol{\tau}(X))$, where $\mathbf{C}(X, t) = \mathbf{D}\chi^T(X, t)\mathbf{D}\chi(X, t)$ and

$$\boxed{\boldsymbol{\sigma}(X, t) = 2\rho\mathbf{D}\chi(X, t)\frac{\partial \hat{W}}{\partial \mathbf{C}}(\mathbf{C}(X, t), \boldsymbol{\tau}(X))\mathbf{D}\chi^T(X, t).}$$

A well posed problem (hopefully)

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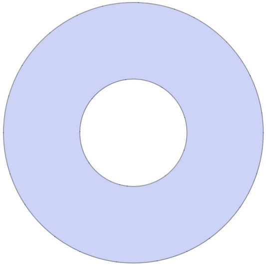
- What remains is to specify the dependance of $\hat{W}$ on $\mathbf{C}$ and $\boldsymbol{\tau}$...

Squeezing a Donut

Without the residual stress $\boldsymbol{\tau}$ we know that



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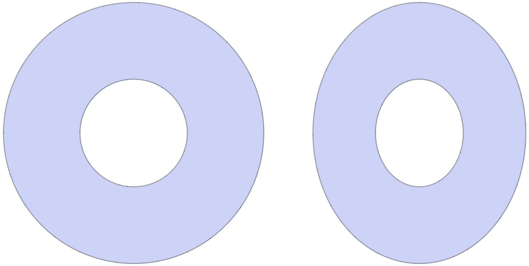

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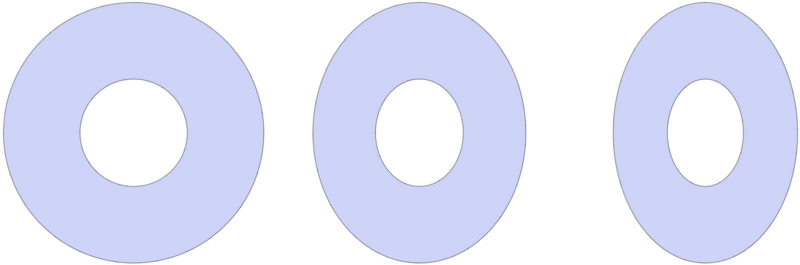

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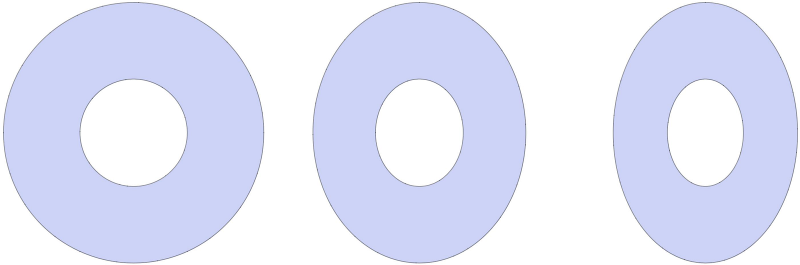

$\longrightarrow$ Potential energy $\hat{W}(\mathbf{C}, \mathbf{0})$ increases $\longrightarrow$

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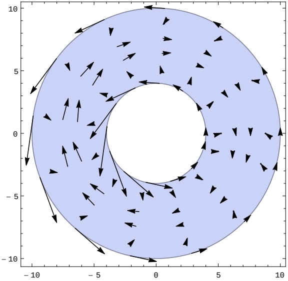
For a residually stressed body it is not so clear



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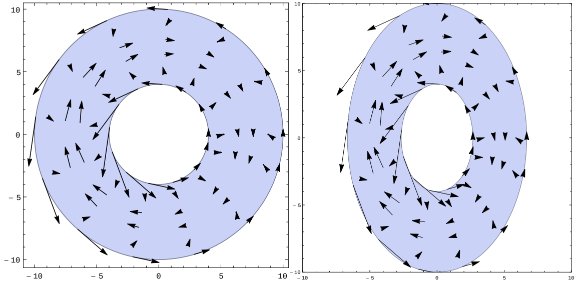

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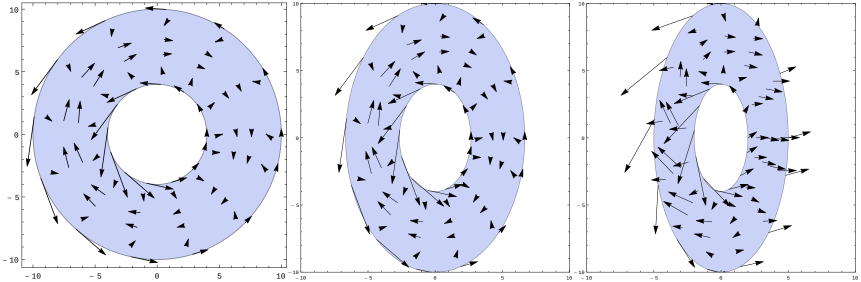

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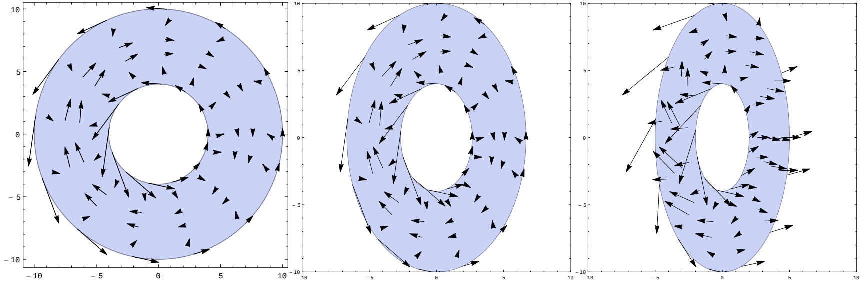

$\longrightarrow$ how does the potential energy $\hat{W}(\mathbf{C}, \boldsymbol{\tau})$ change? $\longrightarrow$

Isotropic Invariants

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$$\hat{W}(\mathbf{C}, \mathbf{0}) = \hat{W}(\mathbf{Q}^T \mathbf{C} \mathbf{Q}, \mathbf{0}), \quad (\mathbf{C} = \mathbf{D}\chi^T \mathbf{D}\chi)$$

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$$\hat{W}(\mathbf{C}, \mathbf{0}) = \hat{W}\left(\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \mathbf{0}\right) = \Psi(\text{tr } \mathbf{C}, (\text{tr } \mathbf{C})^2 - \text{tr } \mathbf{C}^2, \det \mathbf{C}).$$

where

$$\begin{aligned} \text{tr } \mathbf{C} &= \lambda_1 + \lambda_2 + \lambda_3, \\ (\text{tr } \mathbf{C})^2 - \text{tr } \mathbf{C}^2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 - 3 \\ \det \mathbf{C} &= \lambda_1\lambda_2\lambda_3. \end{aligned}$$

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One successful expansion

$$\hat{W}(\mathbf{C},\mathbf{0}) = C_1\text{tr }\mathbf{C} + C_2((\text{tr }\mathbf{C})^2 - \text{tr }\mathbf{C}^2) + C_3\det\mathbf{C}.$$

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One possible set of independent invariants [Shams et al. (2011)] is

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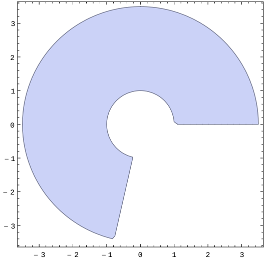
A Geometric Fix



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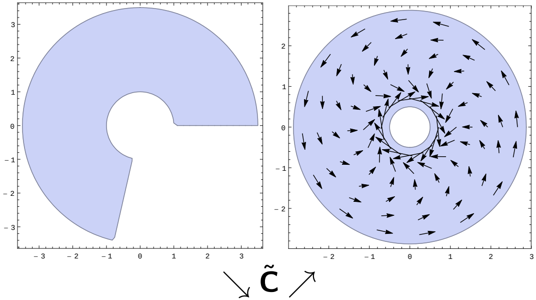

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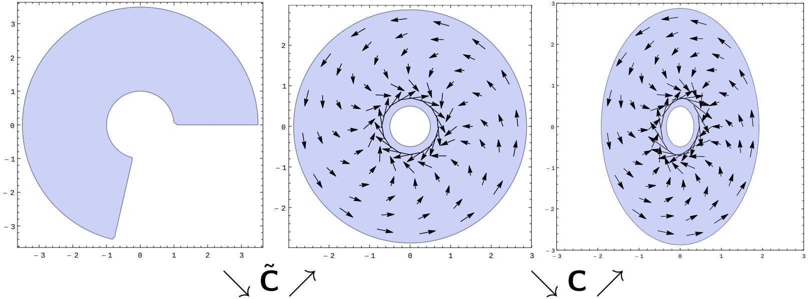

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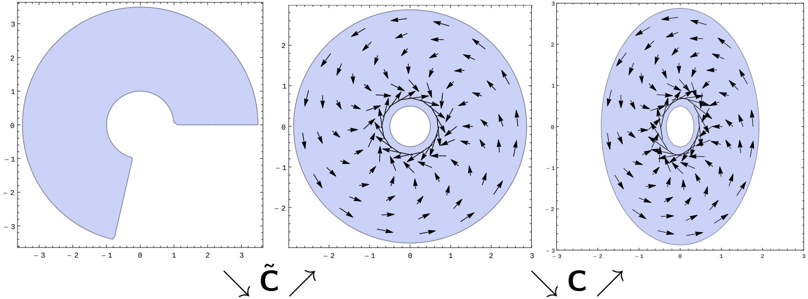

- So we get energy as $W(\mathbf{C}, \boldsymbol{\tau}) = W_C\left(\tilde{\mathbf{C}} \cdot \mathbf{C}\right)$!

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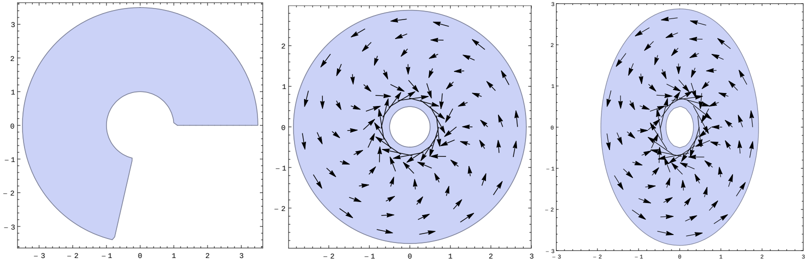
$\downarrow$ Cut and check.



$\searrow \tilde{\mathbf{C}} \nearrow$ $\qquad$ $\searrow \mathbf{C} \nearrow$

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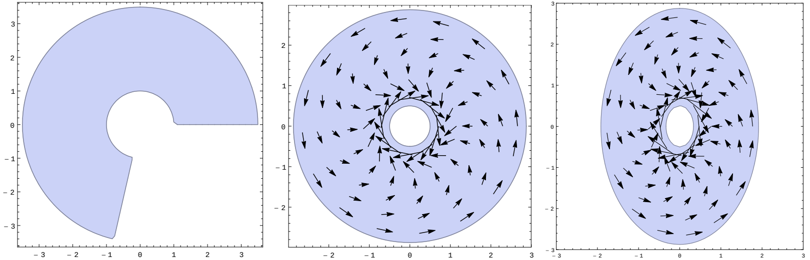

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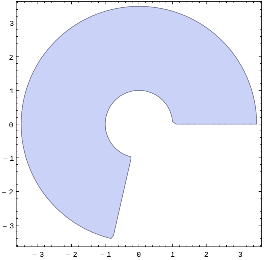
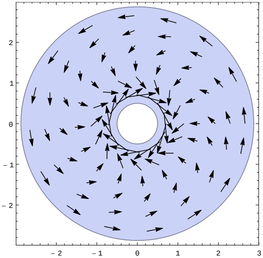
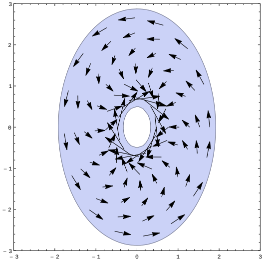


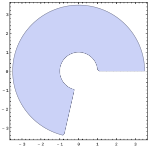




$\searrow \tilde{\mathbf{C}} \nearrow$ $\searrow \mathbf{C} \nearrow$

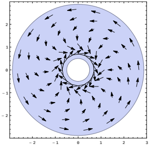
- So we get energy as $W(\mathbf{C}, \boldsymbol{\tau}) = W_C\left(\tilde{\mathbf{C}} \cdot \mathbf{C}\right)$!

A Geometric Fix



$\longrightarrow$

$\tilde{\chi}(R, \Theta)$



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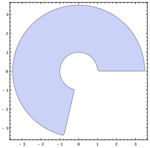

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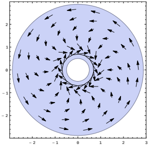
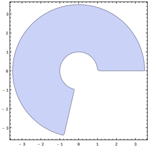


$$A \leq R \leq B$$
$$0 \leq \Theta \leq \Theta_0$$

A Geometric Fix



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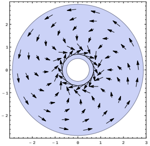
$$\xrightarrow{}$$

$$\tilde{\chi}(R, \Theta)$$

$$\shortparallel$$

$$(r(R), \Theta(\theta))$$



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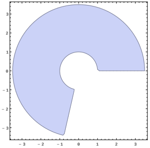

$$A \leq R \leq B$$
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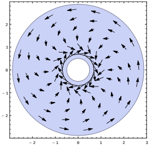

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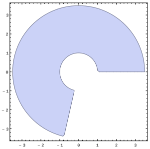
A Geometric Fix



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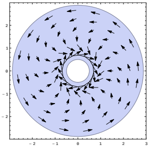

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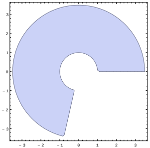
Incompressibility means

$$\det \mathbf{D}\tilde{\chi} = 1 \implies \det \begin{pmatrix} r_R & 0 \\ 0 & \theta_\Theta \end{pmatrix} = 1$$

A Geometric Fix



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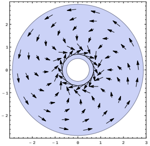

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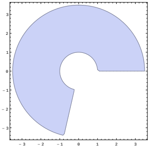

$A \leq R \leq B$

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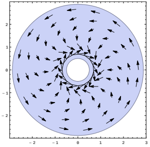
$$\xrightarrow{\hspace{2cm}} \quad \tilde{\chi}(R,\Theta) = (r(R), \Theta(\theta))$$



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$$\implies \quad \theta(\Theta) = \frac{2\pi}{\Theta_0}\Theta \text{ and } r(R) = \sqrt{a^2 + \frac{\Theta_0}{2\pi}(R^2 - A^2)}$$

A Geometric Fix

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implying that

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_{rr} & 0 \\ 0 & \tau_{\theta\theta} \end{pmatrix} = \begin{pmatrix} r_R^2(R) - p(R) & 0 \\ 0 & \theta_\Theta^2(\Theta)/R^2 - p(R) \end{pmatrix},$$

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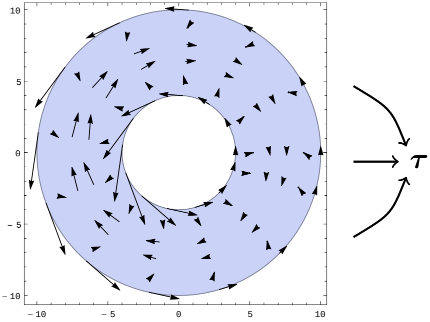
$\tau_{rr}(A) = r_R^2(A) - p(A) = 0 = \tau_{rr}(B) = r_R^2(B) - p(B)$.

Plug into equilibrium equation

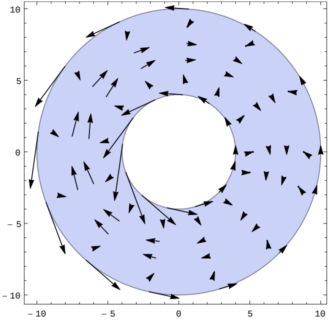
$$\text{div } \boldsymbol{\tau} = 0 \implies r/r_R\,\partial_R\tau_{rr} + \tau_{rr} - \tau_{\theta\theta} = 0,$$

solve for $p(R)$.

Stress τ as Input

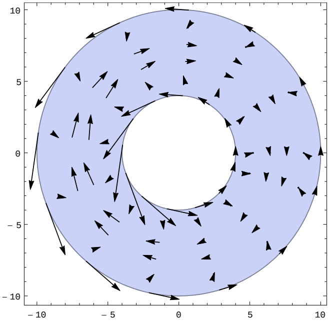


Stress $\boldsymbol{\tau}$ as Input



$$\longrightarrow \boldsymbol{\tau} \longrightarrow \tilde{\mathbf{B}}(\boldsymbol{\tau}) = \mathbf{D}\chi\mathbf{D}\chi^{T}$$

Stress $\boldsymbol{\tau}$ as Input



$$\longrightarrow \boldsymbol{\tau} \longrightarrow \tilde{\mathbf{B}}(\boldsymbol{\tau}) = \mathbf{D}\chi\mathbf{D}\chi^T$$

Adopting

$$W(\mathbf{C}, \boldsymbol{\tau}) = W_C(\tilde{\mathbf{C}}\mathbf{C}) = \frac{\mu}{2}\Big(\text{ tr }(\tilde{\mathbf{C}}\mathbf{C}) - 3\Big) = \frac{\mu}{2}\Big(\text{ tr }(\mathbf{B}\tilde{\mathbf{B}}(\boldsymbol{\tau})) - 3\Big).$$

(remember $\mathbf{C} = \mathbf{D}\chi^T\mathbf{D}\chi$).

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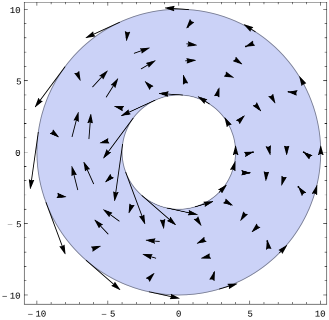

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(remember $\mathbf{C} = \mathbf{D}\chi^T\mathbf{D}\chi$).

- Though we assume there is a virtual stress-free state, that gives $\tilde{\mathbf{C}}$, we don't know what it looks like!

Stress $\boldsymbol{\tau}$ as Input

To find $\tilde{\mathbf{B}}(\boldsymbol{\tau})$:

$$\boldsymbol{\tau} = \mu\rho\mathbf{D}\tilde{\chi}\frac{\partial \mathrm{tr}\ \tilde{\mathbf{C}}}{\partial \tilde{\mathbf{C}}}\mathbf{D}\tilde{\chi}^{T} - p\mathbf{I} = \mu\tilde{\mathbf{B}} - p\mathbf{I},$$

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Assume $\tilde{\mathbf{B}}$ incompressible, then after some algebra an inversion is possible.

Stress $\boldsymbol{\tau}$ as Input

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Assume $\tilde{\mathbf{B}}$ incompressible, then after some algebra an inversion is possible. Plain strain example:

$$\tilde{\mathbf{B}} = \frac{1}{\mu}\boldsymbol{\tau} + \frac{1}{2\mu}\left(-\mathrm{tr}\,\boldsymbol{\tau} + \sqrt{4\mu^2 + (\mathrm{tr}\,\boldsymbol{\tau})^2 - 4\det\boldsymbol{\tau}}\right)\mathbf{I},$$

Stress $\boldsymbol{\tau}$ as Input

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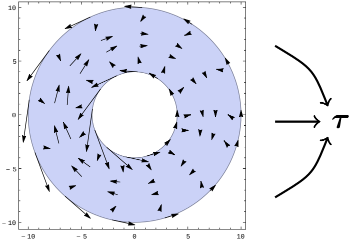
then using $W(\mathbf{C},\boldsymbol{\tau}) = W_C(\mathbf{B}\tilde{\mathbf{B}})$, leading to

$$W(\mathbf{C},\boldsymbol{\tau}) = \frac{1}{2}\mathrm{tr}(\mathbf{B}\boldsymbol{\tau}) + \frac{1}{4}\mathrm{tr}\mathbf{B}\left(-\mathrm{tr}\boldsymbol{\tau} + \sqrt{4\mu^2 + (\mathrm{tr}\ \boldsymbol{\tau})^2 - 4\det\boldsymbol{\tau}}\right) - \mu$$

Stress τ as Input

Looking for a "cut" stress-free state:



Stress $\boldsymbol{\tau}$ as Input

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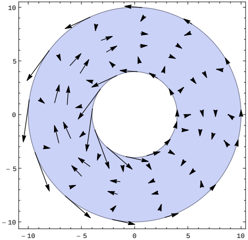

$$\longrightarrow \boldsymbol{\tau} - \tilde{\mathbf{B}}^{-1} \longrightarrow$$

Stress $\boldsymbol{\tau}$ as Input

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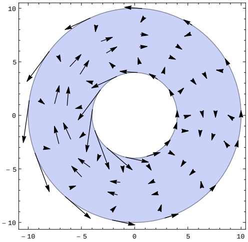

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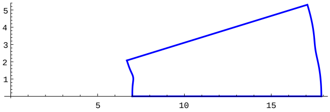

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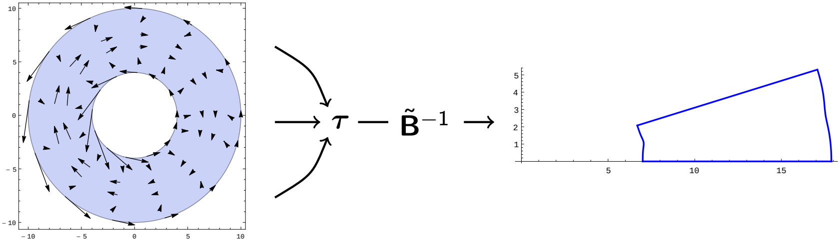

For a given stress field $\boldsymbol{\tau}$

finding a stress-free state embedded in $\mathbb{R}^3$ is like finding a needle in a nine dimensional haystack.

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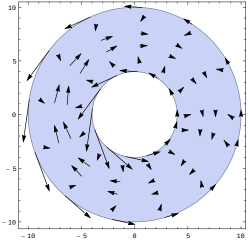

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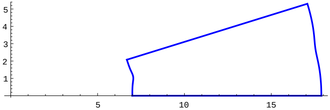

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finding a needle in a nine dimensional haystack.

- Is it necessary that the vitural state be an embedding in $\mathbb{R}^3$?

Where to now?

- This method takes a measured $\boldsymbol{\tau}$ and produces $W(\mathbf{C}, \boldsymbol{\tau})$.

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- Should we only except $\tilde{\chi}$ where the stress-free state is embedded in $\mathbf{R}^3$?

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Any questions?

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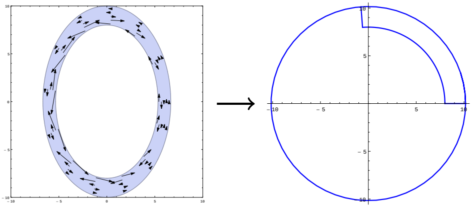

Any questions?
Thanks for listening and hope you enjoyed the talk!

M. Shams and M. Destrade and R.W. Ogden, Initial stresses in elastic solids: Constitutive laws and acoustoelasticity, Wave Motion, 48 (2011) 552 – 567.

Jose Merodio, Ray W. Ogden, Javier Rodriguez, The influence of residual stress on finite deformation elastic response, International Journal of Non-Linear Mechanics, 56 (2013), 43 – 49.