Commensurations and Subgroups of Finite Index of Thompson's Group F

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We determine the abstract commensurator $\mathrm{Com}(F)$ of Thompson's group F and describe it in terms of piecewise linear homeomorphisms of the real line. We show $\mathrm{Com}(F)$ is not finitely generated and determine which subgroups of finite index in F are isomorphic to F. We also show that the natural map from the commensurator group to the quasi-isometry group of F is injective.

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Introduction

Thompson's groups have been extensively studied since their introduction by Thompson in the 1960s, despite the fact that Thompson's account [7] appeared only in 1980. They have provided examples of infinite finitely presented simple groups, as well as some other interesting counterexamples in group theory (see for example, Brown and Geoghegan [3]). Cannon, Floyd and Parry [4] give an excellent introduction to Thompson's groups where many of the basic results used below are proven carefully.

Automorphisms for Thompson's group F were studied by Brin [2], where a key theorem by McCleary and Rubin [6] is used to realize each automorphisms as conjugation by a piecewise linear map. Here, we generalize from automorphisms to commensurations, which are isomorphisms between two subgroups of finite index. These form a group (under a natural equivalence relation involving passing to smaller yet still finite-index subgroups), called the commensurator group.

We classify finite-index subgroups of F, and then we extend Brin's results from automorphisms to commensurations, again realizing every commensuration as conjugation by a piecewise linear homeomorphism of the real line. These maps exhibit a particular structure, satisfying an affinity condition in the neighborhood of ∞ which we use to find the algebraic structure of the commensurator of F.

Commensurators have proven to be an effective tool for investigating quasi-isometries of a group to itself, and for effectively analyzing rigidity, particularly of lattices. In the case of F, the only quasi-isometries of F known previously were automorphisms. This paper provides a wide array of examples of quasi-isometries, since all commensurations are quasi-isometries, and we prove in Section 5 that the commensurator group embeds into the quasi-isometry group in the case of F.

Our approach is algebraic, but we note that elements of the commensurator of F can be represented by marked, infinite, eventually periodic, binary tree pair diagrams. We also note that recently Bleak and Wassink [1] have independently described the finite-index subgroups of F, using different methods.

The paper is organized as follows. In Section 1 we give the necessary definitions, and in Section 2 the first basic results for the finite-index subgroups of F. In Section 3 the main result about the commensurator is stated and proved, and in Section 4 its algebraic structure is given. The proof of the embedding of the commensurator group into the quasi-isometry group is given in Section 5.

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1 Definitions

Let P denote the group of all homeomorphisms f from $\mathbb{R}$ to itself that

(1) are piecewise linear with a discrete (but possibly infinite) set of breakpoints (discontinuities of the derivative of f),

(2) use only slopes that are integral powers of 2,

(3) have their breakpoints in the set $\mathbb{Z}[\frac{1}{2}]$ and

(4) satisfy $f(\mathbb{Z}[\frac{1}{2}]) \subset \mathbb{Z}[\frac{1}{2}]$.

It is easy to check that each element f of P actually satisfies $f(\mathbb{Z}[\frac{1}{2}]) = \mathbb{Z}[\frac{1}{2}]$ and that P has a subgroup of index two which contains only the order preserving elements. We denote this subgroup by P_+. The quotient P/P_+ is generated by the image of the homeomorphism $\tau\colon\ t \mapsto -t$.

Let $f \in P$. We call f *integrally affine* if $f(t) = \varepsilon t + p$ for some integer p and $\varepsilon \in \{\pm 1\}$. We say f is *periodically affine* if $f(t+p) = f(t) + q$ for some non-zero $p, q \in \mathbb{R}$ and *integrally periodically affine* if p and q are integers. Note that all integrally affine maps are integrally periodically affine with $q = \pm p$ depending on whether f is in P_+ or not.

When $\mathcal{P}$ is any of the above properties, then we call f *eventually* $\mathcal{P}$ if f satisfies $\mathcal{P}$ for all $t \in \mathbb{R}$ with $|t| > M$ for some $M > 0$; here $|t|$ denotes the absolute value of t. For example, $f \in P_+$ is eventually integrally affine if there exist $l, r \in \mathbb{Z}$, $M \in \mathbb{R}$, $M > 0$, so that $f(t) = t + r$ for all $t > M$ and $f(t) = t + l$ for all $t < -M$. Notice that l and r may well be different.

It is well-known that Thompson's group F is isomorphic to the subgroup of P_+ consisting of all eventually integrally affine elements (see [4]). It is easy to see that the commutator subgroup F' of F consists of all eventually trivial elements of P_+(those where eventually $f(t) = t$). This group is denoted by $BPL_2(\mathbb{R})$ by Brin [2], where B stands for bounded support.

2 Finite-index Subgroups of F

Let f be an element of F. Since f is eventually integrally affine, there are two integers l, r and a real number $M > 0$ such that $f(t) = t + r$ for $t > M$ and $f(t) = t + l$ for $t < -M$. The two numbers l and r are precisely the two components of the image of f in $\mathbb{Z} \times \mathbb{Z}$ under the abelianization map. The subgroups of finite index of F are in one-to-one correspondence with those of its abelianization $\mathbb{Z} \times \mathbb{Z}$ by the following result.

Proposition 2.1 *Let H be a subgroup of F of finite index. Then H contains F', the commutator subgroup of F, and hence H is normal in F. Moreover, $H' = F'$.*

Proof Since F is finitely generated, H has only finitely many conjugates in F and the intersection of all of them, K say, is normal and of finite index in F. We consider $K \cap F'$, which is thus normal and of finite index in F'. Hence, since F' is simple and infinite, we conclude that $K \cap F' = F'$ and $F' \subset K \subset H$.

Hence H is normal in F. The final claim follows from the fact that H' is contained in F' but also characteristic in H and hence normal in F, whence $F' \subset H'$. □

From this fact we deduce that the finite-index subgroups of F are in bijection with those of $\mathbb{Z} \times \mathbb{Z}$. There is a distinguished family among these—the subgroups $p\mathbb{Z} \times q\mathbb{Z}$. We denote by $[p, q]$, $p, q \in \mathbb{Z}$, the preimage in F under the abelianization homomorphism of the subgroup $p\mathbb{Z} \times q\mathbb{Z}$ of $\mathbb{Z} \times \mathbb{Z}$. Thus $F = [1, 1]$ and $F' = [0, 0]$.

3 The Commensurator Group

As mentioned before, a *commensuration* of a group G is an isomorphism $\alpha\colon A \to B$, where A and B are subgroups of G of finite index. Two commensurations α and β are equivalent if they agree on some subgroup of finite index in G. In view of this, the product $\beta \circ \alpha$ of two commensurations

$$\alpha\colon A \to B \quad \text{and} \quad \beta\colon C \to D$$

is defined on $\alpha^{-1}(B \cap C)$. The set of all commensurations of G modulo the above equivalence relation, together with this composition, forms a group called the *commensurator of* G which we denote by $\mathrm{Com}(G)$. If G is a subgroup of the group H, then the (relative) commensurator of G in H, $\mathrm{Com}_H(G)$, consists of all elements h of H for which $G \cap G^h$ has finite index in both G and G^h; here $G^h = h^{-1}Gh$.

The main result of this paper is the following.

Theorem 3.1 *The commensurator of F is isomorphic to $\mathrm{Com}_P(F)$, which consists of all eventually integrally periodically affine elements (of P).*

The strategy of the proof is to find a large group where F is a subgroup, and in such a way that every commensuration can be seen as a conjugation by an element of the large group. The group P plays this role in the case of F.

In order to explain this strategy, we need some definitions and one of the main results of McCleary and Rubin [6]. Let $(L, <)$ be a dense linear order. By *interval* we mean a nonempty open interval. A subgroup G of $\mathrm{Aut}(L)$ is *locally moving* if for every interval I there exists a nontrivial element $g \in G$ which acts as the identity on $L \setminus I$. Finally, G is *n-interval-transitive* if for every pair of sequences of intervals $I_1 < \cdots < I_n$ and $J_1 < \cdots < J_n$ there exists $g \in G$ such that $I_k^g \cap J_k \neq \varnothing$ for $1 \leq k \leq n$. Below, $\overline{L}$ denotes the Dedekind completion of L which is assumed to have no endpoints.

Theorem 3.2 (McCleary–Rubin [6]) *Assume $(L_i, <)$ is a dense linear order without endpoints and let $G_i \subset \mathrm{Aut}(L_i)$ be locally moving and 2-interval transitive, $i = 1, 2$. Suppose that $\alpha\colon\ G_1 \to G_2$ is an isomorphism. Then there is a monotonic bijection $\tau\colon\ \overline{L}_1 \to \overline{L}_2$ which induces α, that is, $g^\alpha = \tau^{-1} g \tau$ for every $g \in G_1$; and τ is unique.*

Being locally moving and having 2-interval transitivity are local properties in the sense that a group inherits these from any of its subgroups.

Proof of Theorem 3.1 View $\mathbb{Z}[\frac{1}{2}]$ as a dense linear order and F as the eventually integrally affine elements of P_+. Let $\alpha\colon\ A \to B$ be a commensuration of F. By Proposition 2.1, both A and B contain F' which is (obviously) locally moving and 2-interval transitive (see [2, Lemma 2.1]). So Theorem 3.2 tells us that α is induced by conjugation with a unique element of $\mathrm{Homeo}(\mathbb{R})$. This yields an injective homomorphism $\Psi\colon\ \mathrm{Com}(F) \to \mathrm{Homeo}(\mathbb{R})$.

Next, we show that the image of Ψ is in fact contained in P. By Proposition 2.1, each commensuration of F induces an automorphism of F'. In other words, the image of Ψ is contained in $N_{\mathrm{Homeo}(\mathbb{R})}(F')$, the normalizer of F' in $\mathrm{Homeo}(\mathbb{R})$. But this normalizer is equal to P by Theorem 1 of Brin [2]. The existence and uniqueness statements in Theorem 3.2 now imply that Ψ is an isomorphism between $\mathrm{Com}(F)$ and $\mathrm{Com}_P(F)$, which proves the first part of Theorem 3.1.

Let $\alpha \in \mathrm{Com}(F)$ and choose positive integers p and q so large that α is defined on the subgroup $[p,q]$, that is $[p,q]^\alpha$, the image of $[p,q]$ under α, is contained in F. By what was said above, we can view α as conjugation by an element of P. So for $f \in [p,q]$ we find $f^\alpha = \alpha^{-1} f \alpha$ to be eventually integrally affine. Suppose for a moment that α is order preserving and that $f(t) = t + kq$ for $t \gg 0$, where $k \in \mathbb{Z}$. Then

$$f^\alpha(t) = (\alpha \circ f \circ \alpha^{-1})(t) = \alpha(f(\alpha^{-1}(t))) = \alpha(\alpha^{-1}(t) + kq) = t + r$$

must hold for some $r \in \mathbb{Z}$. In other words, $\alpha^{-1}(t+r) = \alpha^{-1}(t) + s$ for some integers r and s and all $t \gg 0$. Since f was arbitrary, we may assume that $k \neq 0$, which implies that $s \neq 0$, and hence also $r \neq 0$. Therefore α^{-1}, and hence α, must be integrally periodically affine near infinity. A similar calculation holds for $t \ll 0$ and also when α is order reversing. Consequently, each commensuration of F must be eventually integrally periodically affine.

It remains to show that each eventually integrally periodically affine $\beta \in P$ induces a commensuration of F by conjugation. Suppose $\beta(t+p) = \beta(t) + q$ for $t \gg 0$ and $\beta(t+p') = \beta(t) + q'$ for $t \ll 0$, with $p, q, p', q' \in \mathbb{Z} \setminus \{0\}$. Let $U = [p', p]$ if β is order preserving and set $U = [p, p']$ otherwise. Then for $f \in U$, we have

$$f^{\beta}(t) = \begin{cases} \beta(\beta^{-1}(t) + kp) = t + kq, & t \gg 0 \\ \beta(\beta^{-1}(t) + k'p') = t + k'q', & t \ll 0 \end{cases}$$

where $k, k' \in \mathbb{Z}$ depend on f. Together with a similar argument for β^{-1} one easily sees that $U^{\beta} = [q', q]$ or $[q, q']$, depending on whether β is order preserving or not. Theorem 3.1 is thus established. □

We immediately obtain the following corollaries from this result.

Corollary 3.3 *A subgroup U of F of finite index is isomorphic to F if and only if $U = [p, q]$ for some positive integers p and q.*

Proof Suppose U is a subgroup of finite index in F. If U is isomorphic to F, then there exists an eventually integrally periodically affine $\alpha \in P$ with $F^{\alpha} = U$ and calculations as above show that U must be of the form $[p, q]$. On the other hand, the final paragraph of the proof of the theorem read with $p = p' = 1$ shows that $[q', q]$ is isomorphic to F for every choice of positive integers q and q'. This completes the proof. □

Finally, since each subgroup of finite index in F contains $[p, q]$ for some positive integers p and q by Proposition 2.1, we have the following results.

Corollary 3.4 *Every finite-index subgroup of F is virtually F.*

Corollary 3.5 *A group is commensurable with F if and only if it is a finite extension of F.*

4 The Structure of $\mathrm{Com}(F)$

Descriptions of elements of $\mathrm{Com}(F)$ as conjugations in P allow us to study its structure as a group. An element α of $\mathrm{Com}(F)$ is eventually integrally periodically affine, so there exist positive integers p, p', q, q' and a real number M such that

$$\alpha(t + p) = \alpha(t) + q, \text{ for } t > M$$
$$\alpha(t + p') = \alpha(t) + q', \text{ for } t < -M.$$

We need a lemma about affine functions, whose proof is elementary and left to the reader.

Lemma 4.1 *Let $f\colon\ \mathbb{R} \to \mathbb{R}$ be an integrally periodically affine map, and assume that there are integers i, i', j, j' such that for all $t \in \mathbb{R}$ we have*

$$f(t+i) = f(t) + j \quad \text{and} \quad f(t+i') = f(t) + j'.$$

Then we have

$$f(t+r) = f(t) + s,$$

where
$$r = \gcd(i, i') \quad \text{and} \quad s = \gcd(j, j').$$

Furthermore, we have

$$\frac{i}{j} = \frac{i'}{j'}.$$

From this lemma, we see that the integers p, p', q, q' for element of $\mathrm{Com}(F)$ depend only on the element.

We recall that $\mathrm{Com}(F)$ has a subgroup of index 2, denoted $\mathrm{Com}^+(F)$, formed by the commensurations induced by conjugations by piecewise-linear maps which preserve the orientation of $\mathbb{R}$.

Proposition 4.2 *There exists a surjective homomorphism $\Phi\colon\ \mathrm{Com}^+(F) \to \mathbb{Q}^* \times \mathbb{Q}^*$ defined by*

$$\Phi(f) = \left(\frac{p}{q}, \frac{p'}{q'}\right).$$

Here $\mathbb{Q}^*$ denotes the multiplicative group of the positive rational numbers.

The map is obviously well-defined due to the lemma above, and it is very easy to see that it is a homomorphism of groups. The two components of the map capture the behavior at both ends, eventually near $-\infty$ and eventually near $+\infty$. The two numbers p/q and p'/q' measure the "rate of growth" of the map at both ends.

A corollary of this result is that, as expected, $\mathrm{Com}(F)$ is infinitely generated.

5 Commensurations as Quasi-isometries

Let G be a finitely generated group. Quasi-isometries of G can be naturally composed, and there is a natural notion of equivalence class of quasi-isometries. Two quasi-isometries are considered equivalent if they are a bounded distance apart in the sense that f and g are considered equivalent if there exists a number $M > 0$ such that $d(f(t), g(t)) \leq M$ for all t in G.

Equivalence classes of quasi-isometries form elements of the group of quasi-isometries $QI(G)$ of G. It is well known that the commensurator group admits a map to the quasi-isometry group, since all commensurations give maps between finite index subgroups which are canonically quasi-isometric to the ambient group. The result we want to prove in this section is that for Thompson's group F, this map is one-to-one.

Theorem 5.1 *The natural homomorphism* $\mathrm{Com}(F) \to QI(F)$ *is injective.*

We begin with an elementary lemma.

Lemma 5.2 *Given an element $\tau \in P$ which is different from the identity, there exist two intervals I and J of the real line, whose endpoints are dyadic integers, with $\tau(I) = J$, and such that $I \cap J = \varnothing$.*

Proof The case when the slope of τ is always 1 or -1 is trivial. For a map $t \mapsto t + k$ has a small interval (of length less than k) whose image is disjoint from it. If $\tau = -Id$ the result is trivial.

If the slope is not constantly equal to 1, it has a piece with slope $\pm 2^i$ with $i \neq 0$. Assume without loss of generality (by possibly taking τ^{-1} instead of τ) that $i > 0$. Hence there are two intervals $[a, b]$ and $[c, d]$ such that $\tau(a) = c$ and $\tau(b) = d$ and also $d - c = 2^i(b - a)$. It is possible that $[a, b]$ and $[c, d]$ overlap, but since $[c, d]$ is much larger than $[a, b]$ (at least twice the size), we can choose as J a small interval inside $[c, d]$ which is disjoint from $[a, b]$. By construction, the preimage I of J is in $[a, b]$, and hence I and J are disjoint. □

Proof of Theorem 5.1 We now take a nontrivial $\tau \in \mathrm{Com}(F)$. By the previous lemma, there exist intervals I and J satisfying the conditions stated above and, in addition, that I, and hence J, have endpoints of the form $k/2^j$ and $(k+1)/2^j$. We consider all elements of F whose support (that is, the part where they are not the identity) is contained in I. Those elements form a subgroup which is isomorphic to F itself. Let f be one such element. Since its support is inside I, its image under the commensuration τ, that is, $f^\tau = \tau \circ f \circ \tau^{-1}$, has support inside J.

Hence, the distance (inside F) from f to f^τ is given by the distance from the identity to the element $f^\tau f^{-1}$. But this element has its support inside the disjoint union $I \cup J$, and the two parts are independent from each other (one given by f and the other one by f^τ). By work of Cleary and Taback [5], this subgroup—elements with support in $I \cup J$ which is a direct product of two clone subgroups in their terminology—is quasi-isometrically embedded in F. Hence, we can take elements f_n with support inside I with arbitrarily large norm, and hence $f_n^\tau f_n^{-1}$ has also arbitrarily large norm. This proves that the image of τ, a quasi-isometry, is not at bounded distance from the identity and the proof is complete. □

References

[1] **C Bleak**, **B Wassink**, *Finite index subgroups of R. Thompson's group F*Preprint, arXiv:0711.1014

[2] **M G Brin**, *The chameleon groups of Richard J. Thompson: automorphisms and dynamics*, Inst. Hautes Études Sci. Publ. Math. 84 (1996) 5–33 (1997)

[3] **K S Brown**, **R Geoghegan**, *An infinite-dimensional torsion-free* FP_∞ *group*, Invent. Math. 77 (1984) 367–381

[4] **J W Cannon**, **W J Floyd**, **W R Parry**, *Introductory notes on Richard Thompson's groups*, Enseign. Math. (2) 42 (1996) 215–256

[5] **S Cleary**, **J Taback**, *Geometric quasi-isometric embeddings into Thompson's group F*, New York J. Math. 9 (2003) 141–148 (electronic)

[6] **S McCleary**, **M Rubin**, *Locally Moving Groups and the Reconstruction Problem for Chains and Circles*Preprint, arXiv:math.LO/0510122

[7] **R J Thompson**, *Embeddings into finitely generated simple groups which preserve the word problem*, from: "Word problems, II (Conf. on Decision Problems in Algebra, Oxford, 1976)", Stud. Logic Foundations Math. 95, North-Holland, Amsterdam (1980) 401–441

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