Addendum to "Commensurations and Subgroups of Finite Index of Thompson's Group F"

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It is shown that the abstract commensurator of F is composed of four building blocks. Two isomorphism types of simple groups, the multiplicative group of the positive rationals and a cyclic group of order two. The main result establishes the simplicity of a certain group of piecewise linear homeomorphisms of the real line.

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The purpose of this note is to extend earlier work [2], where we described the commensurator group of Thompson's group F. We prove that an interesting subgroup of $\mathrm{Com}(F)$ is simple and describe the algebraic structure of $\mathrm{Com}(F)$ in terms of short exact sequence of simple groups and the multiplicative group of the positive rationals. For all the details and notation, see the paper [2].

1 The group of eventually periodic maps

Previously [2] we described the commensurator group of F as the group of the eventually integrally periodically affine maps in P, which is defined in Section 1 of [2]. These elements may preserve or reverse the orientation of the real line. We also showed that the index-two subgroup $\mathrm{Com}^+(F)$ of orientation-preserving maps fits into the short exact sequence

$$1 \longrightarrow K \longrightarrow \mathrm{Com}^+(F) \longrightarrow \mathbb{Q}^* \times \mathbb{Q}^* \longrightarrow 1$$

whose kernel K is exactly those elements f of P_+ for which there exists $M > 0$, and two positive integers p, p' such that

$$f(t+p) = f(t) + p \text{ for } t \geq M \text{ and}$$
$$f(t+p') = f(t) + p' \text{ for } t \leq -M.$$

Now we can associate to each element $f \in K$ two integrally periodically affine maps f_+ and f_-, which coincide with f near ∞ and $-\infty$, respectively. This property leads to the following definitions.

For $p \in \mathbb{N}$, we denote by H_p the subgroup of P_+ of p-periodically affine maps, that is

$$H_p = \{f \in P_+ \mid f(t+p) = f(t) + p \text{ for all } t \in \mathbb{R}\}.$$

Clearly, if $p|q$, then $H_p \subset H_q$, whence we define the subgroup H as a direct limit under inclusion by

$$H = \bigcup_{p=1}^{\infty} H_p.$$

The maps f_+ and f_- now give rise to a homomorphism

$$\rho\colon\ K \longrightarrow H \times H,$$

given by $\rho(f) = (f_-, f_+)$. The kernel consists of the eventually trivial elements, and therefore equals F', the commutator subgroup of F (see [1] or [2]). In other words, we get the short exact sequence

$$1 \longrightarrow F' \longrightarrow K \longrightarrow H \times H \longrightarrow 1.$$

Brin [1] showed that $\mathrm{Aut}^+(F) = \rho^{-1}(H_1 \times H_1)$ and established the short exact sequence

$$1 \longrightarrow F \longrightarrow \mathrm{Aut}^+(F) \longrightarrow T \times T \longrightarrow 1,$$

where T is Thompson's group T (see [3]). Since we clearly have a map $H_1 \to T$, due to the fact that a map which is 1-periodically affine can be viewed as a map on the circle S^1 given by $\mathbb{R}/\mathbb{Z}$, an alternative version of this sequence is

$$1 \longrightarrow F' \longrightarrow \mathrm{Aut}^+(F) \longrightarrow H_1 \times H_1 \longrightarrow 1.$$

These two sequences are related by the short exact sequence

$$1 \longrightarrow A_1 \longrightarrow H_1 \longrightarrow T \longrightarrow 1,$$

whose kernel A_1 is the maps $t \mapsto t+k$ for integers k. Clearly A_1 is isomorphic to $\mathbb{Z}$.

It is straightforward to verify that any element α of $\mathrm{Com}^+(F)$ which satisfies $\alpha(t+1) = \alpha(t) + p$ for all $t \in \mathbb{R}$ conjugates H_1 to H_p and A_1 to A_p, the group of maps of the form $t \mapsto t + kp$ with $k \in \mathbb{Z}$. So we clearly have a short exact sequence

$$1 \longrightarrow A_p \longrightarrow H_p \longrightarrow T \longrightarrow 1.$$

We note that this extension is, in fact, central, and that one may view this copy of T as acting on the circle of length p given by $\mathbb{R}/p\mathbb{Z}$. We summarise this discussion as follows.

Theorem 1 *The structure of the group* $\mathrm{Com}(F)$ *and its index-two subgroup* $\mathrm{Com}^+(F)$ *is given by the following short exact sequences and equalities.*

$$1 \longrightarrow \mathrm{Com}^+(F) \longrightarrow \mathrm{Com}(F) \longrightarrow C_2 \longrightarrow 1$$

$$1 \longrightarrow K \longrightarrow \mathrm{Com}^+(F) \longrightarrow \mathbb{Q}^* \times \mathbb{Q}^* \longrightarrow 1$$

$$1 \longrightarrow F' \longrightarrow K \longrightarrow H \times H \longrightarrow 1, \qquad H = \bigcup_{p=1}^{\infty} H_p$$

$$1 \longrightarrow A_p \longrightarrow H_p \longrightarrow T \longrightarrow 1$$

$$A_p \cong \mathbb{Z} \textit{ is central in } H_p$$

2 Simplicity of the group H

Here we exploit the well-known fact that T is simple (eg. [3]) to prove our main result.

Theorem 2 *The group* $H = \{f \in P_+ \mid f(t+p) = f(t)+p \text{ for some } p \in \mathbb{N}\}$ *is simple.*

Note that for $p, q \in \mathbb{N}$ with $p|q$, we have $H_p \subset H_q$ and $A_p \supset A_q$. So the theorem says that in the union H the groups A_p cease to be normal. This is due to the following.

Lemma 3 *A normal subgroup of* H_p *is either* H_p *or it is contained in* A_p.

Proof In the light of the isomorphism between H_p and H_1 which carries A_p to A_1, it suffices to consider the case $p = 1$. Let N be a normal subgroup of H_1 and consider its image in T. Since T is simple, the image of N is either $\{1\}$ or the whole T. If the image is $\{1\}$, then $N \subset A_1$. So we assume that the image is T, which yields the exact sequence

$$1 \longrightarrow B \longrightarrow N \longrightarrow T \longrightarrow 1$$

with kernel $B = N \cap A_1 \subset A_1$. It follows that $B = A_r$ for some r, and we find that

$$H_1/N \cong A_1/A_r \cong \mathbb{Z}/r\mathbb{Z}.$$

In particular H_1/N is abelian. The proof will be complete once we show that H_1 is equal to its commutator subgroup, because then $N = H_1$. In order to establish this, we recall from [3] that T is generated by three elements x_0, x_1 and c subject to the relators

$$[x_0x_1^{-1}, x_0^{-1}x_1x_0], \quad [x_0x_1^{-1}, x_0^{-2}x_1x_0^2], \quad x_1x_0^{-1}cx_1c^{-1},$$
$$(x_0^{-1}cx_1)^2x_0^{-1}c^{-1}, \quad x_1x_0^{-2}cx_1^2x_0^{-1}x_1^{-1}x_0x_1^{-1}c^{-1}x_0 \quad \text{and} \quad c^3.$$

This easily gives rise to a finite presentation for H_1 with three generators x_0, x_1 and c subject to the same relators, except for c^3 which has to be replaced by the two relators $[c^3, x_0]$ and $[c^3, x_1]$. Here x_0, x_1 and c are the preimages of the corresponding generators for T, as defined in [3], with $x_0(0) = x_1(0) = 0$ and $c(0) = -1/4$; composition is then to be read from right to left, as in [3]. In this case c^3 is the map $t \mapsto t-1$ which generates A_1. Modulo the commutator subgroup of H_1, the third, fourth and fifth relators yield the relators $x_0^{-1}x_1^2$, $x_0^{-3}x_1^2c$ and $x_0^{-1}x_1$, respectively, which in turn imply $x_0 = x_1 = c = 1$. This proves that $[H_1, H_1] = H_1$. □

Proof of Theorem 2. Let N be a non-trivial normal subgroup of H. According to Lemma 3, for each p, we have that $N \cap H_p$ is either H_p or it is contained in A_p.

We claim that if $N \cap H_p = H_p$ for some p, then this happens for all $p \in \mathbb{N}$. We take $q \in \mathbb{N}$. Then $N \cap H_{pq}$ is a normal subgroup of H_{pq}, and

$$N \cap H_{pq} \supset N \cap H_p = H_p \supsetneq A_p \supset A_{pq},$$

which shows that $N \cap H_{pq} = H_{pq}$, by the lemma. Thus, in this case N contains all H_q, and hence $N = H$.

The only case left now is that $N \cap H_p \subset A_p$ for all p. Since N is non-trivial and the A_p are infinite cyclic, there exists a p with $N \cap H_p = A_{rp}$ for some $r \geq 1$. But then

$$A_{2pr} \supset N \cap H_{2pr} \supset N \cap H_p = A_{pr} \supsetneq A_{2pr}$$

which is a contradiction. Thus the only normal subgroups of H are H and the identity as claimed. □

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