COMMENSURATIONS AND METRIC PROPERTIES OF HOUGHTON'S GROUPS

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ABSTRACT. We describe the automorphism groups and the abstract commensurators of Houghton's groups. Then we give sharp estimates for the word metric of these groups and deduce that the commensurators embed into the corresponding quasi-isometry groups. As a further consequence, we obtain that the Houghton group on two rays is at least quadratically distorted in those with three or more rays.

INTRODUCTION

The family of Houghton groups $\mathcal{H}_n$ was introduced by Houghton [7]. These groups serve as an interesting family of groups, studied by Brown [2], who described their homological finiteness properties, by Röver [10], who showed that these groups are all subgroups of Thompson's group V, and by Lehnert [9] who described the metric for $\mathcal{H}_2$. Lee [8] described isoperimetric bounds, and de Cornulier, Guyot, and Pitsch [4] showed that they are isolated points in the space of groups.

Here, we classify automorphisms and determine the abstract commensurator of $\mathcal{H}_n$. We also give sharp estimates for the word metric which are sufficient to show that the map from the abstract commensurator to the group of quasi-isometries of $\mathcal{H}_n$ is an injection.

1. DEFINITIONS AND BACKGROUND

Let $\mathbb{N}$ be the set of natural numbers (positive integers) and $n \geq 1$ be an integer. We write $\mathbb{Z}_n$ for the integers modulo n with addition and put $R_n = \mathbb{Z}_n \times \mathbb{N}$. We interpret R_n as the graph of n pairwise disjoint rays; each vertex (i, k) is connected to $(i, k + 1)$. We denote by Sym_n,

FSym_n and FAlt_n, or simply Sym, FSym and FAlt if n is understood, the full symmetric group, the finitary symmetric group and the finitary alternating group on the set R_n, respectively.

The *Houghton group* $\mathcal{H}_n$ is the subgroup of Sym consisting of those permutations that are eventually translations (of each of the rays). In other words, the permutation σ of the set R_n is in $\mathcal{H}_n$ if there exist integers $N \geq 0$ and $t_i = t_i(\sigma)$ for $i \in \mathbb{Z}_n$ such that for all $k \geq N$, $(i,k)\sigma = (i, k + t_i)$; throughout we will use right actions.

Note that necessarily the sum of the translations t_i must be zero because the permutation needs of course to be a bijection. This implies that $\mathcal{H}_1 \cong \mathsf{FSym}$.

For $i, j \in \mathbb{Z}_n$ with $i \neq j$ let $g_{ij} \in \mathcal{H}_n$ be the element which translates the line obtained by joining rays i and j, given by

$$(i,n)g_{ij} = (i, n-1) \text{ if } n > 1,$$
$$(i,1)g_{ij} = (j,1),$$
$$(j,n)g_{ij} = (j, n+1) \text{ if } n \geq 1 \text{ and}$$
$$(k,n)g_{ij} = (k,n) \text{ if } k \notin \{i,j\}.$$

We also write g_i instead of $g_{i\,i+1}$. It is easy to see that $\{g_i \mid i \in \mathbb{Z}_n\}$, as well as $\{g_{ij} \mid i, j \in \mathbb{Z}_n,\ i \neq j\}$, are generating sets for $\mathcal{H}_n$ if $n \geq 3$ as we can simply check that the commutator $[g_0, g_1] = g_0^{-1} g_1^{-1} g_0 g_1$ transposes $(1,1)$ and $(2,1)$. In the special case of $\mathcal{H}_2$, an additional generator to g_0 is needed and we choose τ which fixes all points except for transposing $(0,1)$ and $(1,1)$.

It is now clear that the commutator subgroup of $\mathcal{H}_n$ is given by

$$\mathcal{H}_n' = \begin{cases} \mathsf{FAlt}, & \text{if } n \leq 2 \\ \mathsf{FSym}, & \text{if } n \geq 3 \end{cases}$$

For $n \geq 3$, we thus have a short exact sequence

$$1 \longrightarrow \mathsf{FSym} \longrightarrow \mathcal{H}_n \overset{\pi}{\longrightarrow} \mathbb{Z}^{n-1} \longrightarrow 1$$

where $\pi(\sigma) = (t_0(\sigma), \ldots, t_{n-2}(\sigma))$ is the abelianization homomorphism. We note that as the sum of all the eventual translations must be zero, we have the last translation is determined by the preceding ones:

$$(1) \qquad\qquad t_{n-1}(\sigma) = -\sum_{i=0}^{n-2} t_i(\sigma).$$

We will use the following facts freely throughout this paper, see Dixon and Mortimer [5] or Scott [11].

Lemma 1.1. *The group* FAlt *is simple and equal to the commutator subgroup of* FSym, *and* $\mathrm{Aut}(\mathsf{FAlt}) = \mathrm{Aut}(\mathsf{FSym}) = \mathsf{Sym}$.

2. Automorphisms of $\mathcal{H}_n$

Here we determine the automorphism group of $\mathcal{H}_n$. First we establish that we have to look no further than Sym. We let $N_G(H)$ denote the normalizer, in G, of the subgroup H of G.

Proposition 2.1. *Every automorphism of $\mathcal{H}_n$, $n \geq 1$, is given by conjugation by an element of* Sym, *that is to say* $\mathrm{Aut}(\mathcal{H}_n) = N_{\mathsf{Sym}}(\mathcal{H}_n)$.

Proof. From the above, the finitary alternating group FAlt is the second derived subgroup of $\mathcal{H}_n$, and hence characteristic in $\mathcal{H}_n$. So every automorphism of $\mathcal{H}_n$ restricts to an automorphism of FAlt. Since $\mathrm{Aut}(\mathsf{FAlt}) = \mathsf{Sym}$, this restriction yields a homomorphism $\mathrm{Aut}(\mathcal{H}_n) \longrightarrow \mathsf{Sym}$ and we only need to show that it is injective.

In order to see this let $\alpha \in \mathrm{Aut}(\mathcal{H}_n)$ be an automorphism whose restriction to FAlt is trivial. We let $k \in \mathbb{N}$ and consider the following six consecutive points $a_\ell = (i, k + \ell)$ of R_n for $\ell \in \{0, 1, \ldots, 5\}$.

We denote by g_i^α the image of g_i under α, and by $(x\,y\,z)$ the 3-cycle of the points x, y and z. Using the identities

$$g_i^{-1}(a_1\,a_2\,a_3)g_i = (a_0\,a_1\,a_2) \text{ and } g_i^{-1}(a_3\,a_4\,a_5)g_i = (a_2\,a_3\,a_4)$$

and applying α, which is trivial on FAlt, we get

$$(g_i^\alpha)^{-1}(a_1\,a_2\,a_3)g_i^\alpha = (a_0\,a_1\,a_2) \text{ and } (g_i^\alpha)^{-1}(a_3\,a_4\,a_5)g_i^\alpha = (a_2\,a_3\,a_4),$$

which imply that g_i^α maps a_3 to a_2. Applying a similar argument to all points in the branches i and $i + 1$, it follows that $g_i^\alpha = g_i$, and since i was arbitrary, this means that α is trivial. $\square$

With Lemma 1.1 in mind we now present the complete description of $\mathrm{Aut}(\mathcal{H}_n)$.

Theorem 2.2. *For $n \geq 2$, the automorphism group $\mathrm{Aut}(\mathcal{H}_n)$ of the Houghton group $\mathcal{H}_n$ is isomorphic to the semidirect product $\mathcal{H}_n \rtimes \mathcal{S}_n$, where $\mathcal{S}_n$ is the symmetric group that permutes the n rays isometrically.*

Proof. It is clear that every isometric permutation of the rays, (that is, automorphism of the graph R_n), induces an automorphism of $\mathcal{H}_n$. By the proposition, it therefore suffices to prove that every $\alpha \in \mathsf{Sym}$ which normalizes $\mathcal{H}_n$ must map $(i, k + m)$ to $(j, l + m)$ for some $k, l \geq 1$ and all $m \geq 0$.

To this end, we pick $\alpha \in N_{\mathsf{Sym}}(\mathcal{H}_n)$ and $\sigma \in \mathcal{H}_n$. Since $\sigma^\alpha (= \alpha^{-1}\sigma\alpha)$ is in $\mathcal{H}_n$ and maps the point $x\alpha$ to $(x\sigma)\alpha$, these two points must lie on the same ray for all but finitely many $x \in R_n$. Similarly, x and

$x\sigma$ lie on the same ray for all but finitely many $x \in R_n$, as $\sigma \in \mathcal{H}_n$. Combining these two facts, we see that α maps all but finitely many points of ray i to one and the same ray, say ray j.

Notice that we have the freedom to choose σ such that x and $x\sigma$ are neighbors for all but finitely many x on ray i. If x and $x\alpha$ were not neighbors on ray j for infinitely many x, then $t_j(\sigma^\alpha) > 1$. However, there is simply is not enough room left on ray i so that α^{-1} can map all but finitely many points from ray j to ray i. So $t_j(\sigma^\alpha) = 1$ and the result follows. □

3. Commensurations of $\mathcal{H}_n$

First, we recall that a commensuration of a group G is an isomorphism $A \xrightarrow{\phi} B$, where A and B are subgroups of finite index in G. Two commensurations ϕ and ψ of G are equivalent if there exists a subgroup A of finite index in G, such that the restrictions of ϕ and ψ to A are equal. The set of all commensurations of G modulo this equivalence relation forms a group, known as the (abstract) *commensurator of G*, and is denoted $\mathrm{Com}(G)$. In this section we will determine the commensurator of $\mathcal{H}_n$.

For a moment, we let H be a subgroup of a group G. The *relative commensurator of H in G* is denoted $\mathrm{Com}_G(H)$ and consists of those $g \in G$ such that $H \cap H^g$ has finite index in both H and H^g. Similar to the homomorphism from $N_G(H)$ to $\mathrm{Aut}(H)$, there is a homomorphism from $\mathrm{Com}_G(H)$ to $\mathrm{Com}(H)$; Its kernel consists of those elements of G that centralize a finite index subgroup of H.

In order to pin down $\mathrm{Com}(\mathcal{H}_n)$, we first establish that every commensuration of $\mathcal{H}_n$ can be seen as conjugation by an element of Sym, and then characterize $\mathrm{Com}_{\mathsf{Sym}}(\mathcal{H}_n)$.

Since a commensuration ϕ and its restriction to a subgroup of finite index in its domain are equivalent, we can restrict our attention to the following family of subgroups of finite index in $\mathcal{H}_n$, in order to exhibit $\mathrm{Com}(\mathcal{H}_n)$. For every integer $p \geq 1$, we define the subgroup U_p of $\mathcal{H}_n$ by

$$U_p = \langle \mathsf{FAlt}, g_i^p \mid i \in \mathbb{Z}_n \rangle.$$

We collect some useful properties of these subgroup first, where $A \subset_f B$ means that A is a subgroup of finite index in B.

Lemma 3.1.   (1) *For every p, the group U_p coincides with $\mathcal{H}_n^p$, the subgroup generated by all p^{th} powers in $\mathcal{H}_n$, and hence is characteristic in $\mathcal{H}_n$.*

(2) $U_p' = \begin{cases} \mathsf{FAlt}, & p \text{ even} \\ \mathsf{FSym}, & p \text{ odd} \end{cases}$ and $|\mathcal{H}_n : U_p| = \begin{cases} 2p^{n-1}, & p \text{ even} \\ p^{n-1}, & p \text{ odd} \end{cases}$.

(3) *For every finite index subgroup U of $\mathcal{H}_n$, there exists a $p \geq 1$ with $\mathsf{FAlt} = U_p' \subset U_p \subset_f U \subset_f \mathcal{H}_n$.*

Proof. First we establish (2). We know that $[g_i, g_j]$ is either trivial, when $j \notin \{i-1, i+1\}$, or an odd permutation. So the commutator identities $[ab, c] = [a, c]^b [b, c]$ and $[a, bc] = [a, c][a, b]^c$ imply the first part, and the second part follows immediately using the facts from Section 1.

Part (1) is now an exercise, using that $\mathsf{FAlt}^p = \mathsf{FAlt}$.

In order to show (3), let U be a subgroup of finite index in $\mathcal{H}_n$. The facts that FAlt is simple and U contains a normal finite index subgroup of $\mathcal{H}_n$, imply that $\mathsf{FAlt} \subset U$. Let p be the smallest even integer such that $(p\mathbb{Z})^{n-1}$ is contained in the image of U in the abelianisation of $\mathcal{H}_n$. It is now clear that U_p is contained in U. $\qquad\square$

Noting that $\mathrm{Com}(\mathcal{H}_1) = \mathrm{Aut}(\mathcal{H}_1) = \mathsf{Sym}$, we now characterize the commensurators of Houghton's groups.

Theorem 3.2. *Let $n \geq 2$. Every commensuration of $\mathcal{H}_n$ normalizes U_p for some even integer p. The group $N_p = N_{\mathsf{Sym}}(U_p)$ is isomorphic to $\mathcal{H}_n \rtimes (\mathcal{S}_p \wr \mathcal{S}_n)$, and $\mathrm{Com}(\mathcal{H}_n)$ is the direct limit of N_p with even p under the natural embeddings $N_p \longrightarrow N_{pq}$ for $q \in \mathbb{N}$.*

Proof. Let $\phi \in \mathrm{Com}(\mathcal{H}_n)$. By Lemma 3.1, we can assume that U_p is contained in the domain of both ϕ and ϕ^{-1} for some even p. Let V be the image of U_p under ϕ. Then V has finite index in $\mathcal{H}_n$ and so contains FAlt, by Lemma 3.1. However, the set of elements of finite order in V equals $[V, V]$, whence $[V, V] = \mathsf{FAlt}$, as FAlt and FSym are not isomorphic. This means that the restriction of ϕ to FAlt is an automorphism of FAlt, and hence yields a homomorphism

$$\iota \colon \mathrm{Com}(\mathcal{H}_n) \longrightarrow \mathrm{Com}_{\mathsf{Sym}}(\mathcal{H}_n).$$

That ι is injective follows from a similar argument to the one in Proposition 2.1 applied to $g_i{}^p$ and six points of the form $a_\ell = (i, k + p\ell)$ with $\ell \in \{0, 1, \ldots, 5\}$. Since the centralizer in Sym of FAlt, and hence of any finite index subgroup of $\mathcal{H}_n$, is trivial, the natural homomorphism from $\mathrm{Com}_{\mathsf{Sym}}(\mathcal{H}_n)$ to $\mathrm{Com}(\mathcal{H}_n)$ mentioned above is also injective, and we conclude that $\mathrm{Com}(\mathcal{H}_n)$ is isomorphic to $\mathrm{Com}_{\mathsf{Sym}}(\mathcal{H}_n)$.

From now on, we assume that $\phi \in \mathrm{Com}_{\mathsf{Sym}}(\mathcal{H}_n)$. In particular, the action of ϕ is given by conjugation, and our hypothesis is that $U_p^\phi \subset \mathcal{H}_n$. Now we can apply the argument from the second paragraph in the proof of Theorem 2.2 to $\sigma \in U$ and σ^ϕ (instead of σ^α), which only uses that

both are elements of $\mathcal{H}_n$. So ϕ maps all but finitely many points of ray i to one and the same ray, say ray j. The same argument shows that ϕ^{-1} maps all but finitely many points of ray j to ray i and we conclude that ϕ is a bijection between ray i and ray j after excluding finitely many points.

Since g_i^p has p infinite orbits intersecting ray i, the element $(g_i^p)^\phi$ has, up to finitely many exceptions, p infinite orbits intersecting ray j. Since $(g_i^p)^\phi$ is an element of $\mathcal{H}_n$, we see that $t_j((g_i^p)^\phi) = \pm p$. The same argument applies to ray $i+1$ and almost all of its image under ϕ which shows that $(g_i^p)^\phi \in U_p$. Since i was arbitrary and similar arguments hold for ϕ^{-1}, this means that ϕ normalises U_p.

This proves the theorem, where $\mathcal{S}_p \wr \mathcal{S}_n$ is the permutational wreath product of the symmetric group of degree p, which permutes the residue classes modulo p of one ray isometrically, and the symmetric group of degree n, acting as top group, permuting the n rays isometrically. $\square$

We note that $\mathrm{Com}(\mathcal{H}_n)$ is not finitely generated, for if it were, it would lie in some maximal N_p.

4. Metric estimates for $\mathcal{H}_n$

In this section we will give sharp estimates for the word length of elements of Houghton's groups. This makes no sense for $\mathcal{H}_1$ which is not finitely generated. As mentioned in the introduction, the metric in $\mathcal{H}_2$ was described by Lehnert [9]. In order to deal with $\mathcal{H}_n$ for $n \geq 3$, we introduce the following measure of complexity of an element.

Given $\sigma \in \mathcal{H}_n$, we define $p_i(\sigma)$, for $i \in \mathbb{Z}_n$, to be the largest integer such that $(i, p_i(\sigma))\sigma \neq (i, p_i(\sigma) + t_i(\sigma))$. Note that if $t_i(\sigma) < 0$, then necessarily $p_i(\sigma) \geq |t_i(\sigma)|$, as the first element in each ray is numbered 1.

The *complexity* of $\sigma \in \mathcal{H}_n$ is the natural number $P(\sigma))$, defined by

$$P(\sigma) = \sum_{i \in \mathbb{Z}_n} p_i(\sigma).$$

And the *translation amount* of σ is

$$T(\sigma) = \frac{1}{2} \sum_{i \in \mathbb{Z}_n} |t_i(\sigma)|.$$

The above remark combined with (1) immediately implies $P(\sigma) \geq T(\sigma)$. It is easy to see that an element with complexity zero is trivial, and only the generators g_{ij} have complexity one.

Theorem 4.1. *Let $n \geq 3$ and $\sigma \in \mathcal{H}_n$, with complexity $P = P(\sigma) \geq 2$. Then the word length $|\sigma|$ of σ with respect to any finite generating set*

satisfies

$$P/C \leq |\sigma| \leq KP \log P,$$

where the constants C and K only depend on the choice of generating set.

Proof. Since the word length with respect to two different finite generating sets differs only by a multiplicative constant, we can and will choose $\{\, g_{ij} \mid i, j \in \mathbb{Z}_n, \, i \neq j \,\}$ as generating set to work with, and show that the statement holds with $C = 1$ and $K = 7$.

The lower bound is established by examining how multiplication by a generator can change the complexity. Suppose σ has complexity P and consider σg_{ij}. It is not difficult to see that

$$(2) \quad p_k(\sigma g_{ij}) = \begin{cases} p_k(\sigma) + 1, & \text{if } k = i \text{ and } (i, p_i(\sigma) + 1)\sigma = (i, 1) \\ p_k(\sigma) - 1, & \text{if } k = j, \ (j, p_j(\sigma) + 1)\sigma = (j, 1) \text{ and} \\ & \quad (j, p_j(\sigma))\sigma = (i, 1) \\ p_k(\sigma), & \text{otherwise} \end{cases}$$

where the first two cases are mutually exclusive, as $i \neq j$. Thus $|P(\sigma g_{ij}) - P(\sigma)| \leq 1$, which establishes the lower bound.

The upper bound is obtained as as follows. Suppose $\sigma \in \mathcal{H}_n$ has complexity P. First we show by induction on $T = T(\sigma)$ that there is a word ρ of length at most $T \leq P$ such that the complexity of $\sigma\rho$ is $\bar{P}$ with $\bar{P} \leq P$ and $T(\sigma\rho) = 0$. The case $T = 0$ is trivial. If $T > 0$, then there are $i, j \in \mathbb{Z}_n$ with $t_i(\sigma) > 0$ and $t_j(\sigma) < 0$. So $T(\sigma g_{ij}) = T - 1$. Moreover, $P(\sigma g_{ij}) \leq P$, because the first case of (2) is excluded, as it implies that $t_i(\sigma) = -p_i(\sigma) \leq 0$, contrary to our assumption. This completes the induction step.

We are now in the situation that $\sigma\rho \in \mathsf{FSym}$ and loosely speaking we proceed as follows.

(1) We push all irregularities into ray 0, i.e. multiply by $\prod g_{i0}{}^{p_i(\sigma\rho)}$.
(2) We push all points back into the ray to which they belong, except for those from ray 0 which we mix into ray 1, say.
(3) We push out of ray 1 separating the points belonging to rays 0 and 1 into ray 0 and any other ray, say ray 2, respectively.
(4) We push the points belonging to ray 1 back from ray 2 into it.

These four steps can be achieved by multiplying by an element μ of length at most $4\bar{P}$, such that $\sigma\rho\mu$ is an element which, for each i, permutes an initial segment I_i of ray i. It is clear that μ can be chosen so that $\sum |I_i| \leq \bar{P}$. Finally, we sort each of these intervals using a recursive procedure, modeled on standard merge sort.

In order to sort the interval $I = I_2$ say, we push each of its points out of ray 2 and into either ray 0 if it belongs to the lower half, or to ray 1 if it belongs to the upper half of I. If each of the two halves occurs in the correct order, then we only have to push them back into ray 2 and are done, having used $2|I|$ generators. If the two halves are not yet sorted, then we use the same "separate the upper and lower halves" approach on each of them recursively in order to sort them. In total this takes at most $2|I| \log_2 |I|$ steps.

Altogether we have used at most

$$P + 4\bar{P} + 2 \sum_{i \in \mathbb{Z}_n} |I_i| \log_2 |I_i| \leq 7P \log_2 P$$

generators to represent the inverse of σ; we used the hypothesis $P \geq 2$ in the last inequality. $\square$

We note that because there are many permutations, the fraction of elements which are close to the lower bound goes to zero in much the same way as shown for Thompson's group V by Birget [1] and its generalization nV by Burillo and Cleary [3].

Lemma 4.2. *Let B_k be the set of elements of $\mathcal{H}_n$ with complexity $P \leq k$. The fraction of elements of B_k which have word length greater than $k \log k$ converges exponentially fast to 1.*

Proof. In B_k, we consider the subset of elements with $T = 0$ but which do permute at least one of the points at distance k on one of the rays, so $T = 0$ and $P = k$. There are at least $k!$ elements of this type. Since the number of elements of word length k in any finitely generated group with d generators is at most $(2d)^k$, the relative growth rates of the factorial and exponential functions give us the result. $\square$

5. Subgroup embeddings

We note that each $\mathcal{H}_n$ is a subgroup of $\mathcal{H}_m$ for $n < m$ and that our estimates together with work of Lehnert are enough to give at least quadratic distortion for some of these embeddings.

Theorem 5.1. *The group $\mathcal{H}_2$ is at least quadratically distorted in $\mathcal{H}_m$ for $m \geq 3$.*

Proof. We consider the element σ_n of $\mathcal{H}_2$ which has $T(\sigma_n) = 0$ and transposes $(0, k)$ and $(1, k)$ for all $k \leq n$. Then σ_n corresponds to the word g_n defined in Theorem 8 of [9], where it is shown to have length of the order of n^2 with respect to the generators of $\mathcal{H}_2$ in Lemma 10 there, which are exactly the generators for $\mathcal{H}_2$ given in the introduction. One

easily checks that $\sigma_n = g_{02}{}^n g_{12}{}^n g_{02}{}^{-n} g_{12}{}^{-n}$ in $\mathcal{H}_3$. Thus a family of words of quadratically growing length in $\mathcal{H}_2$ has linearly growing length in $\mathcal{H}_3$, which proves the theorem. $\square$

A natural, but seemingly difficult, question is whether $\mathcal{H}_n$ is distorted in $\mathcal{H}_m$ for $3 \leq n < m$.

6. Some quasi-isometries of $\mathcal{H}_n$

Commensurations give rise to quasi-isometries and are often a rich source of examples of quasi-isometries. Here we show that the natural map from the commensurator of $\mathcal{H}_n$ to the quasi-isometry group of $\mathcal{H}_n$, which we denote by $\mathrm{QI}(\mathcal{H}_n)$, is an injection. That is, we show that each commensuration is not within a bounded distance of the identity. That this is an injection also follows from the more general argument of Whyte which appears as Proposition 7.5 in Farb-Mosher [6].

Theorem 6.1. *The natural homomorphism from $\mathrm{Com}(\mathcal{H}_n)$ to $\mathrm{QI}(\mathcal{H}_n)$ is an embedding for $n \geq 2$.*

Proof. We will show that for each non-trivial $\phi \in \mathrm{Com}(\mathcal{H}_n)$ and every $N \in \mathbb{N}$ we can find a $\sigma \in \mathcal{H}_n$ such that $d(\sigma, \sigma^\phi) \geq N$. By Theorem 3.2, we can and will view ϕ as a non-trivial element of $N_p \subset \mathsf{Sym}$ for some even p.

If ϕ eventually translates a ray i non-trivially to a possibly different ray j, then we let $\sigma = ((i, N)\,(i, N+1))$, a transposition in the translated ray. The image of σ under conjugation by ϕ is the transposition $((j, N+t), (j, N+t'+1))$, and the distance $d(\sigma, \sigma^\phi)$ is the length of $\sigma^{-1}\sigma^\phi$, which is at least N since it moves at least one point at distance N down one of the rays.

If ϕ does not eventually translate a ray but eventually non-trivially permutes ray i with another ray j, then we can show boundedness away from the identity by taking $\sigma = ((j, N)\,(j, N+1))$. The point (i, N) is fixed by σ but is moved to $(i, N+1)$ under σ^ϕ ensuring that the length of $\sigma^{-1}\sigma^\phi$ is at least N.

Finally, if ϕ does not have the preceding two properties, then ϕ is a non-trivial finitary permutation, supported on an interval of size P on ray i and we let $\sigma = g_{i-1}{}^{N+P}$. The image of σ under ϕ is then a finitary permutation involving points at distance at least N in ray i, giving a lower bound of at least N for $d(\sigma, \sigma^\phi)$. $\square$

7. Co-Hopficity

Houghton's groups are long known to be Hopfian although they are not residually finite, see [4]. In this section we will prove that $\mathcal{H}_n$ is not

co-Hopfian, by exhibiting a map which is injective but not surjective. The map is the following:

$$f \colon \mathcal{H}_n \longrightarrow \mathcal{H}_n$$
$$s \mapsto f(s)$$

defined by: if $s(i, n) = (j, m)$, then:

$$f(s)(i, 2n - 1) = (j, 2m - 1) \qquad \text{and} \qquad f(s)(i, 2n) = (j, 2m).$$

It is straightforward to show that f is a homomorphism. It is injective, because if s is not the identity with $s(i, n) \neq (i, n)$, then $f(s)(i, 2n) \neq (i, 2n)$. And clearly the map is not surjective, because the permutation always sends adjacent points $(i, 2n - 1)$, $(i, 2n)$ to adjacent points, and a permutation which does not do this cannot be in the image.

Theorem 7.1. *Houghton's groups $\mathcal{H}_n$ are not co-Hopfian.*

REFERENCES

[1] Jean-Camille Birget. The groups of Richard Thompson and complexity. *Internat. J. Algebra Comput.*, 14(5-6):569–626, 2004. International Conference on Semigroups and Groups in honor of the 65th birthday of Prof. John Rhodes.

[2] Kenneth S. Brown. Finiteness properties of groups. In *Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985)*, volume 44, pages 45–75, 1987.

[3] José Burillo and Sean Cleary. Metric properties of higher-dimensional Thompson's groups. *Pacific J. Math.*, 248(1):49–62, 2010.

[4] Yves de Cornulier, Luc Guyot, and Wolfgang Pitsch. On the isolated points in the space of groups. *J. Algebra*, 307(1):254–277, 2007.

[5] John D. Dixon and Brian Mortimer. *Permutation groups*, volume 163 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1996.

[6] B. Farb and L. Mosher. The geometry of surface-by-free groups. *Geom. Funct. Anal.*, 12(5):915–963, 2002.

[7] C. H. Houghton. The first cohomology of a group with permutation module coefficients. *Arch. Math. (Basel)*, 31(3):254–258, 1978/79.

[8] Sang Rae Lee. Geometry of Houghton's groups, 2012. arXiv:1212.0257.

[9] Jörg Lehnert. Some remarks on depth of dead ends in groups. *Internat. J. Algebra Comput.*, 19(4):585–594, 2009.

[10] Claas E. Röver. *Subgroups of finitely presented simple groups*. PhD thesis, University of Oxford, 1999.

[11] W. R. Scott. *Group theory*. Dover Publications Inc., New York, second edition, 1987.

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