Subgroups of Finitely Presented Simple Groups

Claas H. E. W. Röver

Pembroke College
University of Oxford
Trinity Term 1999

A thesis submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy at the University of Oxford.

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Abstract

This thesis is concerned with the possible structure of subgroups of finitely presented infinite simple groups. We survey the finitely presented simple groups that were known prior to this thesis and prove that they are all torsion locally finite except possibly those for which the conjugacy problem is unsolvable. A group is called torsion locally finite if every finitely generated torsion subgroup is finite.

Then we generalise the old methods for constructing finitely presented simple groups. This, in turn, enables us to describe constructive embeddings of certain recursively presented groups into finitely presented groups which in theory exist by Higman's Embedding Theorem.

What is more, we can construct a class of finitely presented simple groups ${\mathcal{H}_{f,p}}'$ that are not torsion locally finite. More precisely, they have subgroups isomorphic to Grigorchuk-Gupta-Sidki groups which are finitely generated infinite torsion groups under suitable assumptions. We also prove that each group ${\mathcal{H}_{f,p}}'$ is generated by two elements.

In the last two chapters we investigate algorithmic decision problems for the groups $\mathcal{H}_{f,p}$, thereby obtaining a positive answer to the order problem. We also prove that the conjugacy problem is solvable for all elements with 'flat symbols' if p is a prime. This includes all periodic elements, so in particular the conjugacy problem for periodic elements is shown to be solvable. In addition, we describe an effective procedure to decide whether an element has a flat symbol.

We also show that the family ${\mathcal{H}_{f,p}}'$ of finitely presented simple groups contains a countable infinity of isomorphism classes.

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Contents

INTRODUCTION

This work grew out of a desire to get a more detailed picture of the possible structure of subgroups of finitely presented simple groups. The first such groups that come to mind are probably finite simple groups, which obviously have only finite subgroups. Conversely, every finite group can be embedded in some alternating group of sufficiently large degree, for instance, and hence in a finite simple group. This is all we want to say about finite simple groups, and from now on, 'finitely presented simple group' always means 'infinite finitely presented simple group'.

The motivations for our study are numerous. Firstly, simple groups have played an important role in group theory since its inception. Secondly, the class of finitely presented groups is in many ways the most important and natural class of infinite groups (these are fundamental groups of compact manifolds, for example) and has many interesting subclasses, e.g., hyperbolic groups, automatic groups, linear groups over the integers, and Fuchsian groups, to mention just a few. Also very interesting is their connection with algorithmic properties highlighted in the famous embedding theorem of G. Higman [23]: a finitely generated group is embeddable in a finitely presented group if and only if it is recursively presented. To say that a finitely generated group is recursively presented means precisely that there is a recursively enumerable set of defining relations. The set of all relations (words representing the identity) is recursive if and only if the group has a solvable word problem, i.e., there is an algorithm which, given an arbitrary word on the generators, decides whether this word represents the trivial element. This leads to one of the few general results about finitely presented simple groups. Kuznetsov [29] proved that every finitely presented simple group has a solvable word problem. In fact, this is true for every recursively presented simple group. Inspired by Higman's embedding theorem one might seek an algebraic characterisation of groups with solvable word problem. Taking Kuznetsov's result into account, it seems reasonable to pose the following conjecture (see [4]).

Boone-Higman-Conjecture: A finitely generated group has a solvable word problem if and only if it can be embedded in a finitely presented simple group.

We would have loved to prove this conjecture but it appears to be very difficult. However, we do prove several new and related results. Some of them can be summarised as follows.

Theorem A. *There are countably many non-isomorphic 2-generated finitely presented simple groups with finitely generated infinite torsion subgroups.*

As we believe that our results can be more appreciated by seeing them within the present state of knowledge, a quick survey precedes a more detailed discussion of our theorems.

The study of finitely presented simple groups began in 1965 when R. J. Thompson discovered the first two examples (see [40] for a description). These are the groups we call $G_{2,1}$ and $T_{2,1}$. Viewing $G_{2,1}$ as a subgroup of an uncountable group, Thompson [40] proved that a finitely generated group has a solvable word problem if and only if it is a subgroup of a finitely generated simple group which, in turn is a subgroup of a finitely presented group. Moreover, the finitely presented group can be chosen to have a solvable word problem. This is still the most comprehensive result towards the Boone-Higman-Conjecture. Note that there are finitely presented groups with unsolvable word problem, e.g. [32],[3],[31].

In 1974 Higman, upon hearing about Thompson's group, constructed in [24] a countably infinite family of finitely presented simple groups generalising Thompson's $G_{2,1}$. These are the commutator subgroups of the groups called $G_{n,r}$ later in this work. (That name is the same as in [24] and the subscripts are integers $n \geq 2$, $r \geq 1$.) He also showed that this family contains infinitely many isomorphism types. Some time later, in 1987, K. S. Brown discovered a family of finitely presented simple subgroups of the groups $G_{n,r}$ which are generalisations of Thompson's $T_{2,1}$. We will meet these briefly in Chapter 2.

There are several ways to describe these groups. They are automorphism groups of r-generated free algebras in the variety of algebras of sets that are in bijection with their own n-th direct power, cf. [24]. They are also groups of piecewise linear homeomorphisms of the unit interval with prescribed slopes and limited sets of non-differentiable points (see for example [39] and [5]). Furthermore, they are groups of tree diagrams of finite n-ary r-forests. This is the approach in [10]. Yet another point of view, taken in [35] for instance, is their interpretation as groups of maximal inescapable isomorphisms. In this work we use the latter two descriptions, and they are explained in detail in Chapter 1.

The groups $G_{n,r}$ are still not fully understood. For example, the isomorphism problem is not yet known to be solvable in this class of groups. However, Higman's work [24] gives quite some insight. And, amongst many other results, he shows that the group $GL_3(\mathbb{Z})$ of invertible three-by-three matrices over the integers is not isomorphic to a subgroup of any of the groups $G_{n,r}$. Note that $GL_n(\mathbb{Z})$ has a solvable word problem, since multiplication of matrices is effective and only the identity matrix is the trivial element.

His results also imply that the additive group of the rational numbers cannot be embedded in any $G_{n,r}$, and neither can the Baumslag-Solitar groups $BS(l,m)$ which are generated by two elements a and b subject to the single defining relation $b^{-1}a^lb = a^m$ if m divides l.

After studying the obvious generalisations (the groups $\mathcal{G}_{n,r}$ in this work) of Thompson's uncountable supergroup, E. A. Scott [36] showed in 1984 that for any positive integer n there exists a finitely presented simple group with subgroups isomorphic to $GL_n(\mathbb{Z})$. She did this using her method developed in [35] that can be used to embed certain finitely presented groups in finitely presented simple groups: the θ-construction. This is explained in Section 3.3.

The key to these constructions is Thomson's Lemma which says that every subgroup of $\mathcal{G}_{n,r}$ that contains ${G_{n,r}}'$ has a simple commutator subgroup (Lemma 2.3). So, for example, one strategy to obtain finitely presented simple groups is as follows: construct finitely presented supergroups of ${G_{n,r}}'$ inside $\mathcal{G}_{n,r}$ and make sure that their commutator subgroups are of finite index. This is exactly what can be done with the θ-construction, and our constructions also follow these lines.

One of our results (Theorem 4.7) says that every finitely presented simple group obtained with the θ-construction from a torsion locally finite group is also torsion locally finite. We call a group torsion locally finite if every finitely generated torsion subgroup is finite. Putting this together with the observation that each countable locally finite group is a subgroup of any group $G_{n,r}$ (see [24] or Theorem 2.4) we get a complete characterisation of torsion subgroups of $G_{n,r}$: they are precisely the countable locally finite groups. This appears to be new.

A weakness of the θ-construction is that it applies (a priori) to finitely presented groups only. Therefore we develop a generalisation of Scott's method in Chapter 3, which enables us to construct embeddings of certain recursively presented groups into finitely presented simple groups. More precisely, we prove that every Grigorchuk group defined by an almost periodic sequence is isomorphic to a subgroup of some finitely presented simple group (Theorem 4.10). These simple groups are the derived groups of the groups called $\mathcal{H}_{f,p}$ in Chapter 4 where one also finds the necessary definitions concerning Grigorchuk groups.

The finitely presented simple groups ${\mathcal{H}_{f,p}}'$ are interesting for at least two reasons. On the one hand, their constructions are rather rare examples of constructive embeddings of recursively presented groups in finitely presented groups. (It is shown for example in [17] that a non-trivial Grigorchuk group is not finitely presented.) For one specific Grigorchuk group such an embedding was recently obtained by R. I. Grigorchuk [19] using an HNN-extension. In addition, he needs an explicit presentation, whereas our methods do not require a presentation at all. On the other hand, the groups $\mathcal{H}_{f,p}{}'$ are probably the first finitely presented simple, but not torsion locally finite, groups. This is simply because Grigorchuk groups are finitely generated infinite torsion groups for suitably chosen (periodic) defining sequences. Our methods can also be applied to embed other finitely generated infinite torsion groups in finitely presented simple groups, for example, those studied by N. Gupta and S. Sidki in [22] and suitable special groups in the sense of [17]. We also prove that the direct product of any finite family of Grigorchuk-Gupta-Sidki groups whose defining sequences are periodic can be embedded in a finitely presented simple group (Theorem 4.16).

There is one class of previously known finitely presented simple groups that we have not been able to show are torsion locally finite, namely the finitely presented simple groups with unsolvable conjugacy problem described by Scott in [37]. Their construction still relies on Thompson's Lemma, but does not use the θ-construction. Hence our criterion for torsion locally finiteness does not apply, and therefore we cannot assure that the groups $\mathcal{H}_{f,p}{}'$ are new finitely presented simple groups. However, we conjecture that these groups are also torsion locally finite.

One way out of this dilemma would be a solution of the conjugacy problem for $\mathcal{H}_{f,p}{}'$. In such an attempt we have only succeeded in describing a procedure that solves the conjugacy problem for elements with 'flat' symbols (Theorem 6.13) if p is prime. Every periodic element has a flat symbol, so in particular, the conjugacy problem for periodic elements of $\mathcal{H}_{p,f}$ is shown to be solvable. This would be worthless without being able to decide whether a given element has finite order. So we show in Theorem 5.9 that the order problem is solvable for elements of $\mathcal{H}_{f,p}{}'$. In fact we describe a procedure which decides whether an element has a flat symbol. As a by-product we prove that there is a countable infinity of isomorphism types among the groups $\mathcal{H}_{f,p}{}'$ (Theorem 6.14).

Besides all these new results, from a broader point of view, our finitely presented simple groups $\mathcal{H}_{f,p}{}'$ are after all groups of maximal inescapable isomorphisms, and as such not so far away from the groups $G_{n,r}{}'$ and the groups constructed by Scott. Hence, we should mention that recently M. Burger and S. Mozes constructed very different finitely presented simple groups in [9]. These are lattices in direct products of two automorphism groups of regular (non-rooted) trees, and can also be seen as free products with amalgamation of two finitely generated free groups with the amalgamated subgroup having finite index in each factor. In particular, they are torsion-free, and hence not isomorphic to any of the groups $G_{n,r}{}'$ or $\mathcal{H}_{f,p}{}'$.

Let us now give an outline of the individual chapters of this work. We fix our notation and embark on some conventions in the first section of Chapter 1. The remainder of that chapter is a complete introduction to groups of maximal inescapable isomorphism including the language of symbols and tree diagrams. The basic properties of maximal inescapable isomorphisms are gathered there as well.

Chapter 2 presents a selection of known results about the groups $G_{n,r}$ with an emphasis on simplicity (Section 2.1) and the possible structure of subgroups. It contains a proof that $G_{n,r}$ is generated by two elements and some unpublished, though maybe known, results about subgroups. In particular, we show that the Houghton groups are subgroups of every $G_{n,r}$.

Mathematics' technical side shows itself in Chapter 3. First we recall the main features of Scott's paper [35] and describe the θ-construction. Then we introduce fake expansibility as a generalisation of Scott's expansibility, and prove a criterion for finite presentability of certain groups of maximal inescapable isomorphisms, namely fake-E-expansible groups, where E is a finitely presented group.

The applications of this result, in particular the construction of the groups $\mathcal{H}_{f,p}$ and $\mathcal{H}_{f,p}{}'$ and the proof that they are generated by two elements occupy most of Chapter 4. First, however, we record the results of Scott and prove that the θ-construction preserves torsion locally finiteness.

Chapter 5 is a not-so-short but painless description of the algorithm that solves the order problem for $\mathcal{H}_{f,p}$. And finally, Chapter 6 concerns the solvability of the conjugacy problem for elements with flat symbols, in particular periodic elements, of $\mathcal{H}_{f,p}$. An affirmative answer is obtained by a rather long and detailed analysis of different kinds of specialised symbols if p is a prime.

At the end of this work we have included an index of all the major definitions. The reason for this is that we prefer to give definitions in the main text. Since the defined terms are emphasised, readers should have no difficulty in finding them using the index.

Equations are numbered inside each chapter, e.g., the equation labelled (3.4) is the fourth equation in Chapter 3. Theorems, propositions, lemmas, and corollaries are all treated as 'proclaimed statements' and also numbered within chapters, e.g., Lemma 1.7 is the seventh statement in Chapter 1 but not necessarily the seventh lemma of that chapter. References always give both numbers, either in round brackets if they point to an equation, or as in 'Proposition 3.4', if a statement is the target.

Chapter 1

Preliminaries

In this chapter we introduce the main objects of study and fix some notation that is used throughout this thesis. Section 1.1 deals with basic group and set theoretic notions and notational conventions. The remaining sections form an elementary introduction to groups of maximal inescapable isomorphisms which are the main objects of study in this work. Apart from Section 1.5 which is more or less borrowed from [10], most of that material can be found in [35] with minor changes in the notation. We delay definitions of more specialised terms to the places where they are needed.

1.1 Conventions and Notation

We write $\mathbb{Z}, \mathbb{N}$, and $\mathbb{N}_0$ for the integers, the non-negative integers, and the the positive integers, respectively. In particular, $0 \notin \mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By $|X|$ we denote the cardinality of a set X and $X \setminus Y$ is defined as $\{x \in X \mid x \notin Y\}$.

For group elements g and h we write g^h for $h^{-1}gh$, the conjugate of g by h, and let $[g,h]$ denote the commutator $g^{-1}h^{-1}gh$ of g and h. The derived or commutator subgroup of the group G is denoted G' and defined as the subgroup generated by $\{\, [g,h] \mid g,h \in G \,\}$, which is to be read as 'the set of all $[g,h]$ such that g and h are elements of G'. Consequently, we write G'' for the commutator subgroup of the derived subgroup of G. For a subset Y of G the subgroup of G generated by Y is denoted $\langle Y \rangle$ and we also use the standard shorthand notations, e.g., $G' = \langle [g,h] \mid g,h \in G \rangle$. The identity element of a group and the trivial group, as well as the integer 'one', are all denoted by 1, confusion being prevented by the context. If g is a group element, we denote by $|g|$ its order, and we call elements of finite order also periodic or torsion elements. A torsion or periodic group is one with torsion elements only.

If the group G acts on a set X then x^g denotes the image of $x \in X$ under $g \in G$. That is, all our actions are right actions and we use exponential notation; thus, interpreted as an operator, gh means 'g followed by h' and conjugation is one example. Frequently we will be dealing with permutations

π of $\{a_1, a_2, \ldots, a_n\}$, a set with n elements, and it is convenient to interpret π also as a permutation of the subscripts; thus a_i^π and a_{i^π} are equal. As it is slightly odd to have superscripts inside subscripts, we will, in such circumstances, depart from the convention above and write $a_{i\pi}$ in place of a_{i^π}.

To be on the safe side and to avoid superfluous explanations later, we agree to distinguish the *direct product* $\mathsf{X}_{i \in T} U_i$ and the *restricted direct product* $\bigoplus_{i \in I} U_i$ of a family $\{U_i\}_{i \in I}$ of groups, indexed by a set I, where the former is the set of all functions $f : I \longrightarrow \bigcup_{i \in I} U_i$ with $i^f \in U_i$ under pointwise composition, and the latter is the subgroup of those functions with $i^f = 1$ for all but finitely many $i \in I$.

Let us finally recall the definition of some variants of wreath products. Suppose G and H are groups acting on sets X respectively Y. Then the *permutational wreath product*, denoted $G \wr H$, of G with H is the semidirect product of $\mathsf{X}_{y \in Y} G_y$ with H, where each G_y is isomorphic to G and H permutes the components according to its action on Y. If no actions are specified we assume that G and H act on themselves via right multiplication (this is called the standard wreath product in the literature). We will only need the case where Y (or H in standard wreath products) is finite, and it is then convenient to denote elements of $G \wr H$ as $(g_{y_1}, \ldots, g_{y_r})h$, where $g_{y_i} \in G (= G_{y_i})$ and $h \in H$. The important rule to remember is

$$(g_{y_1}, \ldots, g_{y_r})^h = (g_{y_1 h^{-1}}, \ldots, g_{y_r h^{-1}}),$$

where as before, $y_i h$ means y_i^h.

1.2 Inescapable Bases and Subspaces

Let $n, r \in \mathbb{N}$, $n \geq 2$, let W_n be the set of all finite words over the alphabet $\mathcal{A}_n = \{a_1, a_2, \ldots, a_n\}$ including the empty word which we denote by $\emptyset$, and let X_r be a set of r distinct symbols $x_1, \ldots, x_r$ disjoint from $\mathcal{A}_n$. If U and V are sets of words we define $UV = \{uv \mid u \in U,\, v \in V\}$ where uv denotes the concatenation of u and v. We also use the inductive definition $U^{n+1} = U^n U$, $n \geq 1$, with $U^1 = U$ and, if $U = \{u\}$, then we write u^n for U^n.

The binary relation "*u is a prefix of v*" (written $u \leq v$ or $v \geq u$) equips $X_r W_n$ with a partial order. As usual we write $u < v$ if $u \leq v$ and $u \neq v$. A subset B of $X_r W_n$ is called *independent* if its elements are pairwise $\leq$-incomparable and is said to be an *inescapable basis* if it is maximal among independent subsets. Let B be an inescapable basis, $b \in B$ and define C as $(B \setminus \{b\}) \cup \{ba_i \mid 1 \leq i \leq n\}$. Then C is easily seen to be an inescapable basis, too, and we call C the *simple expansion of B at b* or simply a *simple expansion of B*. Furthermore, we call the inescapable basis D an *expansion of B* if there is a finite family of inescapable bases $B_0, \ldots B_m$ with $B = B_0$,

$D = B_m$, and B_i is a simple expansion of B_{i-1} for $1 \le i \le m$. Let us agree to write $B \preceq D$ or $D \succeq B$ if D is an expansion of B. Note that every finite inescapable basis is an expansion of the inescapable basis X_r.

An *inescapable subspace* of $X_r W_n$ is a subset of the form $U = BW_n$, where B is an inescapable basis. The term shall reflect the facts that $UW_n \subset U$ and for every $v \in X_r W_n$ there exists $w \in W_n$ such that $vw \in U$. In fact, these two conditions can also be used to define an inescapable subspace. If B is finite then one says that U is a *finitely based* inescapable subspace. Furthermore, if U is an inescapable subspace we denote by B_U its inescapable basis; this is the set of minimal elements in the partially ordered set U with the induced order from $X_r W_n$. It should be clear that $U \longrightarrow B_U$ is a bijection between inescapable subspaces and inescapable bases and that $U' \subset U$ if and only if $B_{U'} \succeq B_U$.

Using the second characterisation of inescapable subspaces it is straightforward that every intersection of finitely many (finitely based) inescapable subspaces is again a (finitely based) inescapable subspace. If $B_1, \dots, B_m$ are inescapable bases then we denote by $\bigsqcup_{i=1}^m B_i$ the inescapable basis of the inescapable subspace $\bigcap_{i=1}^m B_i W_n$ and, if $m = 2$ we also write $B_1 \sqcup B_2$. This is first of all more legible than $B_{\bigcap_{i=1}^m B_i W_n}$ and secondly more convenient, since it only involves inescapable bases. The following lemma records the basic properties of inescapable bases we will need later.

***Lemma* 1.1** *Let B_i, $1 \le i \le m$, be a finite family of inescapable bases and put $B = \bigsqcup_{i=1}^m B_i$. Then the following hold.*

a) $B \succeq B_i$ for $1 \le i \le m$.

b) $B \subset \bigcup_{i=1}^m B_i$.

c) If $w \in B_j \cap BW_n$ for some j, $1 \le j \le m$, then $w \in B$.

Proof. Part *a*) is obvious. We prove *b*). Without loss of generality we may assume $m = 2$. To obtain a contradiction, let $d \in B$ and assume $d \notin B_1 \cup B_2$. By definition $B \subset B_1 W_n \cap B_2 W_n$, so there are $b \in B_1$, $c \in B_2$ and $w, v \in W_n \setminus \{\emptyset\}$ such that $d = bw = cv$. It follows that $w = xa_i$ and $v = ya_i$ for some i, $1 \le i \le n$, and hence $z = bx = cy \in B_1 W_n \cap B_2 W_n$. But $z < d$ which contradicts the assumption that B is the inescapable basis of $B_1 W_n \cap B_2 W_n$. To prove *c*) we may again assume $m = 2$, and we lose nothing by setting $j = 1$. Supposing $w \notin B$, we find $b \in B$ and $\emptyset \ne x \in W_n$ with $w = bx$. In particular $b < w$. But $b < w \in B_1$ implies $b \notin B_1 W_n$, which in turn contradicts $b \in B \subset B_1 W_n \cap B_2 W_n$. The proof of the lemma is now complete.

Sometimes it is useful to think of $X_r W_n$ as the graph with vertex set $V = X_r W_n$ and edge set $E = \{ (u, ua_i) \mid u \in V, 1 \le i \le n \}$ (see Section 1.5 for definitions concerning graphs). Figure 1 below indicates this graph which in fact is a forest.

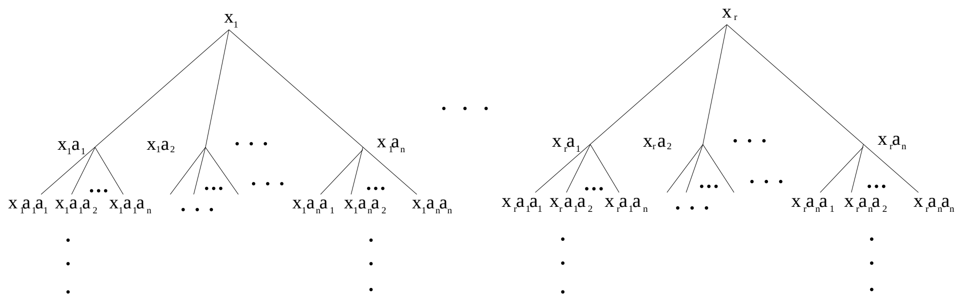


FIGURE 1: THE GRAPH ASSOCIATED WITH X_rW_n

Observe that each vertex v is in a unique connected component which in turn corresponds to a unique element x_i of X_r and that there is a unique reduced path, i.e. a path without backtracking, joining x_i to v. Therefore we may as well identify v with this path. Taking this point of view, an inescapable basis B is identified with the subgraph consisting of all those paths corresponding to the vertices of B. It is now clear that there is a bijection between inescapable bases and those subgraphs which have precisely r connected components each of which contains a unique $x_i \in X_r$ of valency n and all other vertices have valency $n+1$ or 1. In Section 1.5 we will further explore this point of view.

1.3 Groups of Maximal Inescapable Isomorphisms

Recall the convention that maps will be written on the right of their argument and mostly as exponents. Let U and V be inescapable subspaces of X_rW_n. A bijection $\phi : U \longrightarrow V$ is an *inescapable isomorphism (of X_rW_n)* if

$$(uw)^\phi = u^\phi w \text{ holds for all } u \in U, w \in W_n. \tag{1.1}$$

In particular, $u \leq u'$ implies $u^\phi \leq u'^\phi$, where $u, u' \in U$. This shows immediately that the restriction of ϕ to the inescapable basis of U is a bijection $B_U \longrightarrow B_V$. Conversely, for a given bijection $\varphi : B \longrightarrow C$ between inescapable bases B and C there is a unique inescapable isomorphism $\phi : BW_n \longrightarrow CW_n$ whose restriction to B is φ; simply $(bw)^\phi = b^\varphi w$, $b \in B$, $w \in W_n$.

Let $\phi_i : U_i \longrightarrow V_i$, $i = 1, 2$, be inescapable isomorphisms and assume $U_2 \subset U_1$. Then ϕ_1 is called an *extension of* ϕ_2 if the restriction of ϕ_1 to U_2 equals ϕ_2. Note that for every inescapable subspace U which is contained in U_1, U^{ϕ_1} is an inescapable subspace and contained in V_1. An inescapable isomorphism is called *maximal* if it has no proper extension. The following result is Lemma 1 in [35] of which we reproduce a proof for the sake of completeness.

***Lemma* 1.2** *Every inescapable isomorphism has a unique maximalisation.*

Proof. Let $\phi : U \longrightarrow V$ be an inescapable isomorphism of $X_r W_n$. Define U^* to be the set of all $z \in X_r W_n$ for which there exists $y \in X_r W_n$ so that $(zw)^\phi = yw$ for all those $w \in W_n$ with $zw \in U$. Then U^* is an inescapable subspace because $z \in U^*$ clearly implies $zv \in U^*$ for all $v \in W_n$, whence $U^* W_n \subset U^*$, and for all $v \in X_r W_n$ there is $w \in W_n$ with $vw \in U \subset U^*$. For $z \in U^*$ define ϕ^* by $z^{\phi^*} = y$, where y is such that $(zw)^\phi = yw$ for all $w \in W_n$ with $zw \in U$. It is clear from the definition that ϕ^* extends ϕ. We proceed to show that ϕ^* is an inescapable isomorphism. To this end let $z \in U^*$, $v \in W_n$ and suppose $z^{\phi^*} = y$. Then, for all $w \in W_n$ with $zvw \in U$, $(zvw)^\phi = yvw$, whence, by definition of ϕ^*, $(zv)^{\phi^*} = yv = z^{\phi^*} v$. Furthermore, $z^{\phi^*} = z'^{\phi^*} = y$ with $z, z' \in U^*$ implies $(zw)^\phi = (z'w)^\phi = yw$ for all $w \in W_n$ with $zw, z'w \in U$, and hence $zw = z'w$, as ϕ is a bijection. Thus $z = z'$ and ϕ^* is a bijection $U^* \longrightarrow U^{*\phi^*}$. Since $U^{*\phi^*} \supset V$, $U^{*\phi^*}$ is an inescapable subspace. So ϕ^* is an inescapable isomorphism. Suppose now that $\psi : U' \longrightarrow V'$ is an inescapable isomorphism extending ϕ. Then $(uw)^\psi = (uw)^\phi = u^\psi w$ for all $u \in U'$, $w \in W_n$ with $uw \in U$, so $u \in U^*$. Moreover, $u^{\phi^*} = u^\psi$, by definition of ϕ^*, and hence ϕ^* extends ψ. Thus ϕ^* is the unique maximalisation of ϕ and the lemma is proved.

This enables us to define a group structure on the set of maximal inescapable isomorphisms of $X_r W_n$ as follows. Let ϕ_i, $i = 1, 2$, be as above and put $S = V_1 \cap U_2$, $R = S^{\phi_1^{-1}}$ and $T = S^{\phi_2}$. Then it is readily checked that the composition of the restriction of ϕ_1 to R with the restriction of ϕ_2 to S is an inescapable isomorphism $\chi : R \longrightarrow T$, and we define $\phi_1 \phi_2$ to be the (unique) maximalisation of χ.

From now on $\mathcal{G}_{n,r}$ denotes the group of maximal inescapable isomorphisms of $X_r W_n$. It follows from Section 1.2 that the set of maximal inescapable isomorphisms between finitely based inescapable subspaces forms a subgroup of $\mathcal{G}_{n,r}$ which hitherto will be denote by $G_{n,r}$.

Convention. We agree now to call inescapable bases for short *bases* but warn readers that these bases do not share the properties usually expected of bases such as having uniquely determined cardinality etc. However, they are bases in the universal algebra setting of Higman's point of view, cf. [24]. We shall also always identify $X_1 W_n$ with W_n via $x_1 w \mapsto w$.

1.4 SYMBOLS

In this section we describe one way of representing inescapable isomorphisms. Another way is described in Section 1.5. The following approach is already contained implicitly in the work of Thompson [40] and takes the form it is presented here via the works of Higman [24] and Scott [35].

Let us define a *symbol* to be a scheme of the form

$$\begin{pmatrix} b_1 & b_2 & \cdots & b_s \\ g_1 & g_2 & \cdots & g_s \\ c_1 & c_2 & \cdots & c_s \end{pmatrix} \tag{1.2}$$

where the g_i are elements of $\mathcal{G}_{n,1}$ and $\{b_1, b_2, \ldots, b_s\}$ and $\{c_1, c_2, \ldots, c_s\}$ are finite bases of X_rW_n, say B and C, respectively.

Let Γ be the symbol (1.2). Then B and C are called the *top row* and *bottom row* of Γ, respectively, which are denoted by $\operatorname{top}(\Gamma)$ respectively $\operatorname{bot}(\Gamma)$; thus $\operatorname{top}(\Gamma) = \{b_1, b_2, \ldots, b_s\}$ and $\operatorname{bot}(\Gamma) = \{c_1, c_2, \ldots, c_s\}$. Sometimes we call the g_i *middle row entries of* Γ. Note, that although the rows of the symbol Γ appear to be ordered, $\operatorname{top}(\Gamma)$ and $\operatorname{bot}(\Gamma)$ are only defined to be sets. What follows will make clear that the order does not play any role apart from appearing in print. The symbol Γ defines the inescapable isomorphism

$$\begin{array}{rcl} BW_n & \longrightarrow & CW_n \\ b_iw & \longmapsto & c_i(w)^{g_i}, \text{ whenever } w^{g_i} \text{ is defined, } w \in W_n, \end{array}$$

the maximalisation of which we define to be *the element* $\phi \in \mathcal{G}_{n,r}$ *with symbol* Γ.

We also say that $\phi \in \mathcal{G}_{n,r}$ *has the symbol* Δ if Δ is a symbol defining ϕ. Furthermore, for $i \in \{1, 2, \ldots, n\}$,

$$\begin{pmatrix} b_i \\ g_i \\ c_i \end{pmatrix}$$

is called a *column* of the symbol (1.2), and we say that $\phi \in \mathcal{G}_{n,r}$ *has the column* $\mathcal{C}$ or $\mathcal{C}$ *is a column of* ϕ, if ϕ has a symbol Δ so that $\mathcal{C}$ is a column of Δ. It follows directly from the definition, that the order in which the columns of a symbol Δ appear in print does not alter the inescapable isomorphism defined by Δ, so it is perfectly sensible to treat symbols as finite sets of columns and we will do this from time to time. This is why we write $\mathcal{C} \in \Delta$ to denote that $\mathcal{C}$ is a column of Δ and $\operatorname{top}(\mathcal{C})$ and $\operatorname{bot}(\mathcal{C})$ for the top respectively bottom row entry of the column $\mathcal{C}$. The column $\mathcal{C}$ is called *trivial* if it has a trivial middle row entry and $\operatorname{top}(\mathcal{C}) = \operatorname{bot}(\mathcal{C})$. Observe that a symbol defines the trivial element of $\mathcal{G}_{n,r}$ if and only if all its columns are trivial.

A more precise notion is that of an E-*symbol* where E is a subgroup of $\mathcal{G}_{n,1}$, i.e., all the middle row entries are elements of E. This notion will play an important role from Chapter 3 onwards. Note that $G_{n,r}$ consists precisely of those elements which have a 1-symbol (1 denotes the trivial group). This follows from Section 1.2. We also remark that $\mathcal{G}_{n,r}$ embeds in $\mathcal{G}_{n,1}$ for all $n, r \in \mathbb{N}$, showing that $g_i \in \mathcal{G}_{n,1}$ in the definition of a symbol is no serious restriction. For, let B be a finite basis of W_n of cardinality at least r, say $\{b_1, \ldots, b_r\} \subset B$, and let C be a basis of $X_r W_n$. Then by replacing x_i by b_i for $1 \leq i \leq r$ in every element of C we obtain $C' \subset W_n$ and moreover $\hat{C} = C' \cup (B \setminus \{b_1, \ldots, b_r\})$ is a basis of W_n. Finally, if $\phi : CW_n \longrightarrow DW_n$ is an inescapable isomorphism of $X_r W_n$ define $\hat{\phi} : \hat{C}W_n \longrightarrow \hat{D}W_n$ as the identity on $B \setminus \{b_1, \ldots, b_r\}$ and as the obvious map $C'W_n \longrightarrow D'W_n$ induced by ϕ. It is now straightforward to check that $\phi \longmapsto \hat{\phi}$ is an embedding of $\mathcal{G}_{n,r}$ into $\mathcal{G}_{n,1}$. Note, that the image of $G_{n,r}$ under this embedding lies in $G_{n,1}$.

Suppose now Γ is given by (1.2) and g_i has the symbol

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_t \\ h_1 & h_2 & \cdots & h_t \\ v_1 & v_2 & \cdots & v_t \end{pmatrix}.$$

Then it is straightforward to verify that Γ and the symbol

$$\Delta = \begin{pmatrix} b_1 & \cdots & b_{i-1} & b_i u_1 & \cdots & b_i u_t & b_{i+1} & \cdots & b_s \\ g_1 & \cdots & g_{i-1} & h_1 & \cdots & h_t & g_{i+1} & \cdots & g_s \\ c_1 & \cdots & c_{i-1} & c_i v_1 & \cdots & c_i v_t & c_{i+1} & \cdots & c_s \end{pmatrix}$$

define the same maximal inescapable isomorphism. Recall that, by the convention at the end of Section 1.3, $X_1 W_n$ is identified with W_n which is crucial for Δ to be a symbol according to the definition above. We call Δ an *expansion of* Γ *at the column*

$$\begin{pmatrix} b_i \\ g_i \\ c_i \end{pmatrix};$$

if $t = n$ then Δ is called a *simple expansion of* Γ

$$(at \begin{pmatrix} b_i \\ g_i \\ c_i \end{pmatrix}).$$

More generally, we say that the symbol Δ is an expansion of the symbol Σ if there are symbols $\Delta_0 = \Sigma, \Delta_1, \ldots, \Delta_r = \Delta$ and columns $\mathcal{C}_i \in \Delta_i$ such that Δ_{i+1} is an expansion of Δ_i at $\mathcal{C}_i$ for $0 \leq i \leq r-1$. If the symbol Δ is an expansion of the symbol Σ we write $\Delta \succeq \Sigma$ or $\Sigma \preceq \Delta$. As it will be clear from the context whether we talk about symbols or bases, there should be no confusion caused by the multiple use of $\preceq$. For even more clarity, bases will usually be denoted by upper case roman letters and symbols by upper case greek letters. The following easy lemma is Lemma 5 in [35] and it relates symbols to the group structure of $\mathcal{G}_{n,r}$.

***Lemma* 1.3** *If*

$$\Gamma = \begin{pmatrix} u_1 & \cdots & u_s \\ g_1 & \cdots & g_s \\ v_1 & \cdots & v_s \end{pmatrix} \quad \text{and} \quad \Delta = \begin{pmatrix} v_1 & \cdots & v_s \\ h_1 & \cdots & h_s \\ w_1 & \cdots & w_s \end{pmatrix}$$

are symbols for the inescapable isomorphisms ϕ and ψ, respectively, then

$$\begin{pmatrix} u_1 & \cdots & u_s \\ g_1 h_1 & \cdots & g_s h_s \\ w_1 & \cdots & w_s \end{pmatrix}$$

is a symbol for $\phi\psi$.

Let us also record the next result (see [35] Lemma 4) which says that a column for $\phi \in \mathcal{G}_{n,r}$ is already determined by its top row entry. Note that a similar statement holds with bottom row entry instead of top row entry; simply replace ϕ by ϕ^{-1}.

***Lemma* 1.4** *If*

$$\begin{pmatrix} b \\ x \\ c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b \\ y \\ d \end{pmatrix}$$

are both columns of $g \in \mathcal{G}_{n,r}$ then $x = y$ and $c = d$.

Let Γ be the symbol (1.2). Then Γ^{-1} denotes the symbol

$$\begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ g_1^{-1} & g_2^{-1} & \cdots & g_s^{-1} \\ b_1 & b_2 & \cdots & b_s \end{pmatrix},$$

and moreover, if ϕ is the element defined by Γ, then ϕ^{-1} is the element defined by Γ^{-1}, by Lemma 1.3.

For symbols Γ and Δ for ϕ respectively ψ we say that *the combination* $\Gamma\Delta$ *exists* if $\mathrm{bot}(\Gamma) = \mathrm{top}(\Delta)$. In this case $\Gamma\Delta$ is the symbol for $\phi\psi$ given by Lemma 1.3. Similarly, we say that the combination $\Gamma_1 \cdots \Gamma_m$ of the symbols $\Gamma_1, \ldots, \Gamma_m$ exists if $\mathrm{bot}(\Gamma_k) = \mathrm{top}(\Gamma_{k+1})$ for $1 \leq k \leq m-1$ and we identify this combination with the symbol given by repeated application of Lemma 1.3.

Let us focus for a moment on the groups $G_{n,r}$. We have already noted that $\phi \in G_{n,r}$ has a 1-symbol, Δ say. Since the identity element of $G_{n,1}$ has the 1-symbol

$$\begin{pmatrix} a_1 & \cdots & a_n \\ 1 & \cdots & 1 \\ a_1 & \cdots & a_n \end{pmatrix},$$

it follows that for each column $\mathcal{C} \in \Delta$, the simple expansion of Δ at $\mathcal{C}$ exists. These observations suffice to obtain the following lemma which we will use freely throughout this work.

***Lemma* 1.5** *Let $\phi, \psi \in G_{n,r}$ and suppose ϕ has the 1-symbol Δ, then the following hold.*

a) Every expansion Γ of Δ is again a 1-symbol for ϕ and $\mathrm{top}(\Gamma) \succeq \mathrm{top}(\Delta)$ as well as $\mathrm{bot}(\Gamma) \succeq \mathrm{bot}(\Delta)$.

b) For the finite basis B the following hold

(i) If $B \succeq \mathrm{top}(\Delta)$ then ϕ has a 1-symbol Γ with $\mathrm{top}(\Gamma) = B$.

(ii) If $B \succeq \mathrm{bot}(\Delta)$ then ϕ has a 1-symbol Γ with $\mathrm{bot}(\Gamma) = B$

c) If ψ has the 1-symbol Γ, then there are expansions $\hat{\Delta}$ and $\hat{\Gamma}$ of Δ respectively Γ so that the combination $\hat{\Delta}\hat{\Gamma}$ exists. Moreover, $\hat{\Delta}$ and $\hat{\Gamma}$ can be constructed effectively.

As an immediate consequence of Lemma 1.3 and 1.5 c) we see that 1-symbols can be used to compute effectively with elements of the groups $G_{n,r}$ in the sense that for any two elements ϕ and ψ of $G_{n,r}$ there are 1-symbols Γ respectively Δ such that the combination $\Gamma\Delta$ exists, i.e., $\Gamma\Delta$ is a 1-symbol for $\phi\psi$. Since a 1-symbol defines the identity element if and only if it has only trivial columns, we see that the groups $G_{n,r}$ have solvable word problem, i.e., it is effectively decidable whether a given element is trivial.

1.5 Tree Diagrams

Here we describe an alternative way of representing elements of the group $G_{n,r}$, namely tree diagrams. This approach can also be found in [10]. Since the tree diagrams get quite complex for general n and r, we restrict ourselves to the group $G_{2,1}$ and leave it to the reader to draw examples for the other groups. The main purpose of this section is to encourage the reader to draw tree diagrams whenever he fears to get lost in the symbol notation. Unfortunately, the symbol notation does not give such a good intuitive picture of inescapable isomorphisms as tree diagrams but it is mathematically more rigorous which is why we decided to use it for preference in the following chapters.

For our purpose a *graph* is a set V, whose elements are called *vertices*, together with a set E of two-element subsets of V. The elements of E are called *edges*. We say that $v \in V$ and $e \in E$ are *incident* if $v \in e$. A *path* p in a graph is a sequence $v_0, v_1, \ldots, v_r$ of vertices such that $\{v_i, v_{i+1}\}$ is an edge for $0 \le i \le r-1$; this path is said to join v_0 and v_r. The path p is called *trivial* if $r = 0$ and *closed* if $v_0 = v_r$. Furthermore, the path p is said to have a backtracking, if $v_i = v_{i+2}$ for some i. A graph is called *connected* if any two vertices can be joined by a path. Let us mention that there are more complex definitions of a graph, for instance to include multiple edges incident with the same two vertices or loops. Since we are only concerned with forests, we can stick to the simple definition above.

As usual, a graph is called a *tree* if it is connected and has no non-trivial closed path without backtracking. And a graph is a *forest* if each connected component is a tree. Let us define a *finite rooted binary tree* to be a finite tree with precisely one vertex of valency 2, the *root*, and all other vertices having valency 3 or 1. The vertices of valency 1 are called *leaves*. (For the general case one would define finite n-ary r-forests, i.e., graphs with r connected components which in turn are finite n-ary rooted trees.) As indicated at the end of Section 1.2, the set of finite bases of W_2 is in bijection with the set of finite rooted binary trees. Let us agree to draw the root of the tree at the top of the picture of (the geometric realisation of) a rooted binary tree and that for every vertex v the vertices va_1 and va_2 are 'southwest' respectively 'southeast' of v, where 'north' is at the top of the page, as usual. This convention allows us to suppress the labels of the vertices in the diagrams. Figure 2, for example, shows the tree corresponding to the basis $\{a_1a_1a_1, a_1a_1a_2, a_1a_2, a_2a_1, a_2a_2\}$.

Remark. In the general case, we first order the elements of the alphabet $\mathcal{A}_n$ and of X_r according to their subscripts and then order each basis of X_rW_n lexicographically. We then draw tree diagrams so that this order corresponds to the order of the leaves read from left to right.

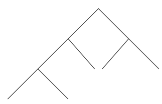


Figure 2

Recall that a finitely based inescapable isomorphism $\phi : S \longrightarrow T$ is uniquely determined by its restriction to B_S and, what is more, $B_S^\phi = B_T$, cf. Section 1.3. Thus we can describe ϕ by drawing the two trees corresponding to B_S and B_T with the leaves labelled in such a way that $b \in B_S$ and $c \in B_T$ have the same label if and only if $b^\phi = c$ and an arrow indicating the range and the domain. We call such an arrangement a *tree diagram.* For example, the element defined by the symbol

$$\Delta = \begin{pmatrix} a_1 & a_2a_1 & a_2a_2 \\ 1 & 1 & 1 \\ a_1a_2 & a_1a_1 & a_2 \end{pmatrix}$$

is the same as the element defined by the tree diagram in Fig. 3.

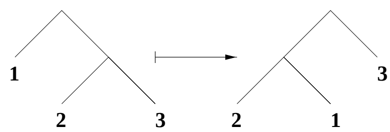


FIGURE 3

If $B_S = B_T$ then ϕ induces a permutation of B_S. In this case we sometimes use a tree diagram with only one tree and arrows indicating the action of ϕ. Such an example is given in Fig. 4 for the element defined by the symbol

$$\Delta = \begin{pmatrix} a_1a_1 & a_1a_2 & a_2 \\ 1 & 1 & 1 \\ a_1a_2 & a_1a_1 & a_2 \end{pmatrix}.$$

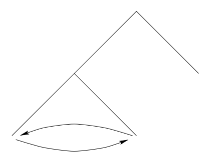


FIGURE 4

Expansions of symbols have an obvious analogue in terms of tree diagrams. Let us omit the details and refer to Fig. 5 instead, which illustrates the computation of the product of the element defined by the tree diagram in Fig. 4 with the element defined by the tree diagram in Fig. 3.

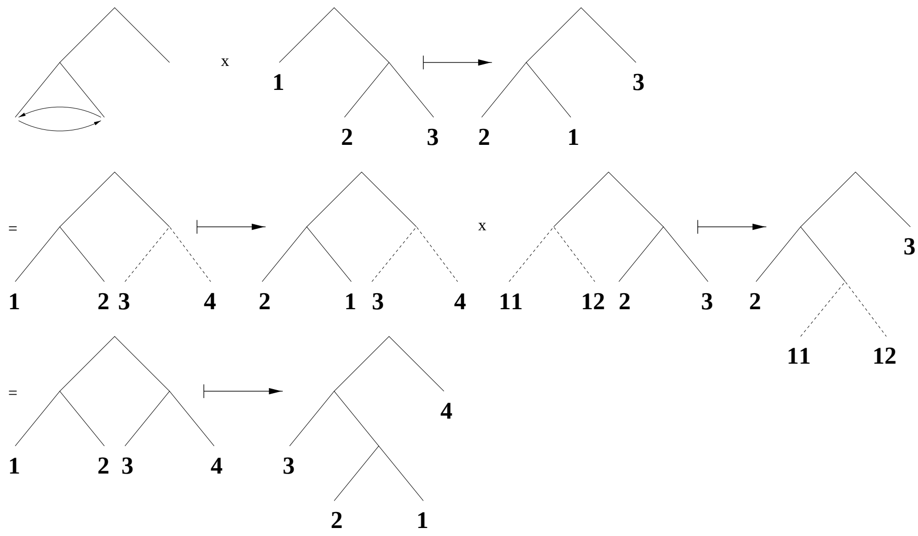


FIGURE 5: MULTIPLYING TREE DIAGRAMS

We conclude this section with an example indicating how we use labelled tree diagrams to represent elements that are given by symbols with non-trivial middle row entries. Let g_1, g_2, g_3 be elements of $\mathcal{G}_{2,1}$. Then Fig. 6 shows the labelled tree-diagram which defines the same element as the symbol

$$\begin{pmatrix} a_1 & a_2a_1 & a_2^2a_1 & a_2^3 \\ g_1 & g_2 & g_3 & 1 \\ a_1a_2a_1 & a_1^2 & a_2 & a_1a_2 \end{pmatrix}.$$

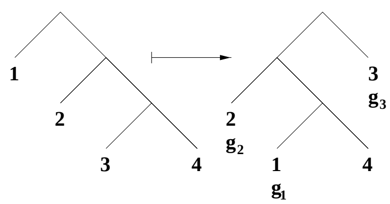


FIGURE 6

Chapter 2

Subgroups: Part I

In this chapter we present a selection of known results about groups of finitely based maximal inescapable isomorphisms. The main focus is on simple groups and what kind of subgroups they may have. In Sections 2.3 and 2.5, however, we obtain at least some unpublished, although maybe well known, results. The final section consists of a proof that each $G_{n,r}$ is generated by two elements. This appears to be new as well.

2.1 Some 'Classical' Subgroups and Simplicity

In this section we define the groups $F_{n,r}$ and $T_{n,r}$ which, for $n = 2$, $r = 1$, were also studied by Thompson [40]. Their generalisations were introduced by Brown [7]. Since we only work inside the groups $G_{n,r}$ in this section, 'symbol' always means '1-symbol'.

Recall that we can equip each finite basis with the lexicographic order as in the remark in Section 1.5, and we write $u < v$ if u comes before v in a lexicon. This convention rules only in this section where the 'prefix order' of Section 1.2 is not used. The first subgroup that comes to mind with this definition is the group $F_{n,r}$ of order preserving elements of $G_{n,r}$. More precisely, $g \in F_{n,r}$ if and only if for all u, u' in the domain of g, $u < u'$ implies $u^g < u'^g$. Equivalently, if Δ is a symbol for g, then the orders of the columns induced by the orders of the top respectively bottom row are the same. It is straightforward to check that this definition is independent of the chosen symbol, so $F_{n,r}$ is really a subgroup. Moreover, $F_{n,r}$ is torsion-free, as can be verified directly. But it also follows from Proposition 4.5, by noting that an element g of $F_{n,r}$ with a symbol Δ satisfying $\text{top}(\Delta) = \text{bot}(\Delta)$ must be trivial.

Let us advance a little and define the group $T_{n,r}$ as the set of all elements of $G_{n,r}$ that preserve the order cyclically. By this we mean that, whenever

$$\begin{pmatrix} u_1 & \cdots & u_s \\ 1 & \cdots & 1 \\ v_1 & \cdots & v_s \end{pmatrix}$$

is a symbol for $g \in T_{n,r}$ with $u_1 < u_2 < \cdots < u_s$, then there is $j \in \{1, \ldots, s\}$ such that $v_j < v_{j+1} < \cdots < v_s < v_1 < \cdots < v_{j-1}$. One can check that this holds for all symbols for g if it holds for one symbol for g, so $T_{n,r}$ is indeed a subgroup. It is clear that $F_{n,r} \subset T_{n,r} \subset G_{n,r}$

Remark. The group $F_{2,1}$ is especially interesting because it appears naturally in various branches of mathematics, e.g. [13],[21]. It has particularly attracted attention in connection with the Day problem concerning amenable groups (for amenable groups see, for example, [11],[12],[41]). We do not want to go into details here, but like to mention that it is not known if $F_{2,1}$ is amenable. Both, an affirmative and a negative answer, would be interesting. If it was amenable, then it would be a finitely presented not elementary amenable group, otherwise it would be a finitely presented non amenable group without non-abelian free subgroups, cf. [10]. However, it seems to be very difficult to decide if $F_{2,1}$ is amenable. All the known criteria do not apply. At this point we should mention that a finitely presented amenable but not elementary amenable group was constructed recently by Grigorchuk in [19], as an HNN-extension of the group G_ω, where $\omega = 012012\cdots$ and $p = 2$ (see Section 4.3 for the definition). Apart from the groups $\mathcal{H}_{f,p}$ and $\mathcal{H}_{f,p}{}'$ of Section 4.4 this is the only finitely presented group having a Grigorchuk group as a subgroup known (to us), and we would like to point out that it is not clear if similar methods work for groups $G_{\omega'}$, where ω' is an arbitrary recursively enumerable sequence. The reason is that the proof uses an explicit presentation of the group G_ω, whereas no explicit presentations are known in the general case. Let us also remark that the group $T_{2,1}$ appears surprisingly in Teichmüller Theory as the universal Ptolémée group, cf. [26].

We proceed with the description of another important subgroup of $G_{n,r}$; its commutator subgroup. It is plain that every element of $G_{n,r}$ can be written as a product of an order preserving element and a permutational element, i.e., an element having a symbol Δ with $\mathrm{top}(\Delta) = \mathrm{bot}(\Delta)$. Such a symbol is called flat. An element is said to be *even* if its associated permutational element has a flat symbol inducing an even permutation in its top row, i.e., the permutation can be written as product of an even number of 2-cycles. We note that for even n every element is even which follows basically from Fig. 7. This figure also indicates that the definition of an even element is independent of the choice of symbol if n is odd.

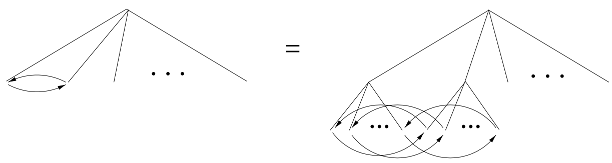


FIGURE 7

It is not difficult to check that the set of even elements is actually a subgroup of $G_{n,r}$, and the following was shown in [24].

***Lemma* 2.1** *The commutator subgroup of $G_{n,r}$ coincides with the set of even elements, in particular, its index is the highest common divisor of 2 and $n-1$.*

The commutator subgroups of $F_{n,r}$ and $T_{n,r}$, as well as the second derived subgroup of the latter, are described in detail in [7] and we do not give these results here. We conclude this section with a result proved by Thompson for $G_{2,1}$, by Higman for $G_{n,r}$, and for $T_{n,r}$ by Brown ([40],[24],[7]). Before we state that result, let us mention that $F'_{n,r}$ is also a simple group, but not finitely generated. In contrast, $F_{n,r}$ is finitely presented.

***Theorem* 2.2** *For all integers $n \geq 2$ and $r \geq 1$, the commutator subgroup of $G_{n,r}$ and the second derived subgroup of $T_{n,r}$ are finitely presented (infinite) simple groups.*

2.2 THOMPSON'S LEMMA

The result of this section could be considered the key to the constructions of more finitely presented simple groups in Chapter 4. It was proved by Thompson [40] for $n = 2$, $r = 1$. Scott adapted the proof for general n and $r = 1$ in [35]. We give a somewhat sketchy proof which we hope is more transparent than the proof in [35], although it follows the same lines. From now on '$<$' denotes again the 'prefix order' defined in Section 1.2.

***Lemma* 2.3** *Every subgroup H of $\mathcal{G}_{n,r}$ which contains the commutator subgroup of $G_{n,r}$ has a simple commutator subgroup, for all $n, r \in \mathbb{N}$, $n \geq 2$.*

Proof. To begin with let ν be a non-trivial element of $\mathcal{G}_{n,1}$. Then there are $w, v \in X_r W_n$ such that $w^\nu = v \neq w$ and $\{w, v\}$ is an independent set. To see this assume that $w^\nu = v \neq w$ but $v < w$, then the basis of the domain of ν contains $w' > v$ with $w' \neq w$, and hence w'^ν cannot be $\leq$-comparable with v, by the definition of inescapable isomorphisms. So w' and w'^ν satisfy the assumptions. If on the other hand $w < v$, then apply the above argument to ν^{-1} with the roles of v and w interchanged.

Only in this proof we use infinite symbols and columns of infinite symbols for elements of $\mathcal{G}_{n,r}$. They can be defined in the same manner as symbols with 'bases' replacing 'finite bases'. Having only finitely many pages we certainly need to treat them as collections of columns. So suppose ν has an infinite symbol consisting of the columns

$$\begin{pmatrix} w \\ 1 \\ v \end{pmatrix} \text{ and } \begin{pmatrix} w_i \\ 1 \\ v_i \end{pmatrix}, \ i \in I,$$

where w and v satisfy the conditions above and I is some index set. Let β be the element defined by the tree-diagram in Fig. 8 and define α to be the element with an infinite symbol having the columns

$$\begin{pmatrix} w \\ \beta \\ w \end{pmatrix} \text{ and } \begin{pmatrix} w_i \\ 1 \\ w_i \end{pmatrix}, \ i \in I.$$

Then $\nu^{-1}\alpha\nu$ has the columns

$$\begin{pmatrix} v \\ \beta \\ v \end{pmatrix} \text{ and } \begin{pmatrix} v_i \\ 1 \\ v_i \end{pmatrix}, \ i \in I.$$

Note that α and $\nu^{-1}\alpha\nu$ are actually different elements of ${G_{n,r}}'$ because they induce even permutations. Thus, $\alpha^{-1}\nu^{-1}\alpha\nu$ is a non-trivial element in $N \cap {G_{n,r}}'$, where N is the normal closure of ν. Using the fact that ${G_{n,r}}'$ is a simple group, we have shown that every non-trivial normal subgroup N of H contains ${G_{n,r}}'$.

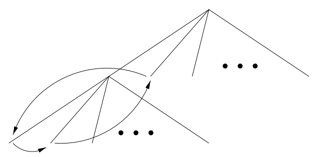


FIGURE 8

Suppose now that H satisfies the hypothesis of the lemma and let N be a non-trivial normal subgroup of H. In the next paragraph we show that H/N is abelian, i.e., $H' \subset N$. This completes the proof by the following argument. Since H' is a group satisfying the hypotheses of the lemma (${G_{n,r}}'$ is non-abelian and simple), each of its non-trivial normal subgroups contains H''. But H'' is normal in H, and therefore equal to H'.

Take two elements, ϕ and χ say, in H. To prove that they commute modulo N, we may, by the above arguments, multiply them with elements of ${G_{n,r}}'$ and show that these altered elements commute. Let us go back to the situation of the first paragraph of the proof. There is certainly an element γ in ${G_{n,r}}'$ with $v^\gamma = w$, so $w^{\nu\gamma} = w$. Hence we can assume that ϕ and χ both have a fixed point. Suppose $y^\chi = y$. Finally, one can now find an element $\delta \in {G_{n,r}}'$ such that $\delta^{-1}\phi\delta$ acts non-trivial on elements in yW_n only. It follows that $\delta^{-1}\phi\delta$ and χ commute, and the proof is complete.

2.3 OBVIOUS SUBGROUPS OF $G_{n,r}$

For a start, let us state explicitly that every finite group is a subgroup of any $G_{n,r}$; simply let it act as a permutation group on a sufficiently large finite basis. Recall that a group is called *locally finite* if every finitely generated subgroup is finite. The following lemma was already established by Higman in [24]. It is proved roughly like this: if the finite group K acts regularly on the independent set S_1, and if S_k is the independent set obtained from S_{k-1} by simply expanding each element of S_{k-1}, then K acts on S_k in the same manner as in its diagonal action on S_1^k, the direct product of k copies of S_1 (S_1^k is not inside X_rW_n, but S_k is). So if K is a subgroup of index k in the group H, then we can let H act regularly on S_k so that the induced action of K viewed as a subgroup of H is precisely that coming from the original action of K on S_1. Iterating this process, each direct limit of countably many finite groups can be embedded in $G_{n,r}$. But a countable locally finite group is exactly such a direct limit.

***Lemma* 2.4** *Every countable locally finite group can be embedded in $G_{n,r}$ for all $n \geq 2$, $r \geq 1$.*

The proof of the next lemma could safely be left as an exercise but the result also follows from the more general Corollary 3.2, by noting that $G_{n,r}$ is 1-expansible.

***Lemma* 2.5** *The class of subgroups of $G_{n,r}$ is closed under countable restricted direct products (see Section 1.1 for the definition) and finite extensions.*

There are also numerous embeddings between the groups $G_{n,r}$ for different choices of n and r, and we only mention a few. For example, the stabiliser of x_1 in $G_{n,r}$ is clearly isomorphic to $G_{n,r-1}$ for $r \geq 2$. We have already shown that $G_{n,r}$ is embeddable in $G_{n,1}$ (Section 1.4). Let us now show that $G_{n,1}$ can be embedded in $G_{2,1}$ for all $n \geq 3$. Define $b_{i+1} = a_2^i a_1$, $0 \leq i \leq n-2$, and $b_n = a_2^{n-1}$. Then $\{b_i \mid 1 \leq i \leq n\}$ is a basis of W_2; denote it by B. For $w \in W_n$ define $w^* \in W_2$ as the element obtained from w by replacing each letter a_i by b_i for $1 \leq i \leq n$. It follows that every basis C of W_n corresponds to a basis C^* in W_2 under this identification, as an easy induction on the number of simple expansions needed to get from $\{a_1, \dots, a_n\}$ to C shows. So given an element g of $G_{n,r}$, with symbol Δ say, we can replace all its top and bottom row entries by their *-images to obtain a symbol Δ^* defining an element g^* of $G_{2,1}$. It is straightforward to check that this defines an embedding. So $G_{2,1}$, although it might look like the 'smallest' group among the $G_{n,r}$ in some sense, contains all the others as subgroups.

There are also some isomorphisms between the groups $G_{n,r}$. For instance, it is not so difficult to see that $G_{n,r}$ is isomorphic to $G_{n,r'}$ whenever r and r' are congruent modulo $n-1$ (see [24]). In [24] Higman proved that $G_{n,r}$ is not isomorphic to $G_{n',r'}$ if $n \neq n'$. However, the isomorphism problem for the groups $G_{n,r}$ is still open; e.g., it is not known if $G_{40,1}$ and $G_{40,7}$ are isomorphic.

The remainder of this section is concerned with the Houghton groups H_m which were introduced by Houghton in [25]. See also [27], [42]. We learned about this group in conversations with J. Harlander and from the paper [7] by Brown which caught our attention with the simplicity results quoted in the first section of this chapter. The main purpose of that paper, however, was to show in a unified way that the groups $F_{n,r}$, $T_{n,r}$, $T''_{n,r}$, $G_{n,r}$, and $G_{n,r}{}'$ are all of type FP_∞ and finitely presented. In fact, that paper also shows this for some other subgroups of $G_{n,r}$, which we do not want to define here. That $F_{n,r}$ is of type FP_∞ was first shown by Brown and Geoghegan in [8]. A group G is said to be of *type* FP_n (resp. FP_∞) if the $\mathbb{Z}G$-module $\mathbb{Z}$ has a projective resolution which is finitely generated in dimensions $\leq n$ (resp. in all dimensions), cf. [6]. Brown shows in [7] that the Houghton group H_m is of type FP_{m-1} but not FP_m and finitely presented if $m \geq 3$.

Before giving some additional remarks, we define the Houghton groups and show that they are embeddable in $G_{n,r}$. This seems to be new. Let $m \geq 1$ be an integer, and define $S = \mathbb{N} \times \{1, \ldots, m\}$. Think of S as the disjoint union of m copies of $\mathbb{N}$, each arranged along a ray emanating from the origin in the plane (see Fig. 9). The Houghton group H_m is the group of all permutations of S which are eventually translations on each ray. More precisely, if $h \in H_m$, then there are integers l_i, $1 \leq i \leq m$, such that

$$(k,i)^h = (k + l_i, i) \tag{2.1}$$

for all sufficiently large $k \in \mathbb{N}$ and all admissible i. Note that $\sum_{i=1}^m l_i$ must be zero, and that H_m contains the finitary symmetric group on the set S for every $m \geq 1$; that is all permutations fixing all but finitely many $s \in S$. For $m \geq 3$, H_m is generated by

$$h_i = \begin{cases} (k,i) \longmapsto (k-1,i), & \text{if } k \geq 2 \\ (k,i+1) \longmapsto (k+1,i+1), & \text{if } k \geq 2 \\ (1,i) \longmapsto (1,i+1) & \\ (k,j) \longmapsto (k,j), & \text{if } j \notin \{i,i+1\} \end{cases}$$

for $1 \leq i \leq m-1$ (see Fig. 9). For $[h_2^{-1}, h_1]$ is the transposition $((1,2),(2,2))$ which together with all its conjugates under the h_i generates the finitary symmetric group F on S, and it is not difficult to see that F and the h_i generate every permutation of S satisfying (2.1).

***Proposition* 2.6** *For all integers $m \geq 1$, the Houghton group H_m is a subgroup of any $G_{n,r}$.*

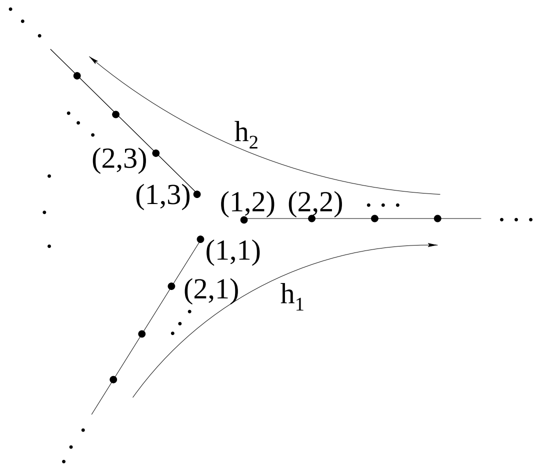


FIGURE 9

Proof. For simplicity we prove this for $G_{2,1}$, only. The general case is similar. It is clear that H_1 is isomorphic to the finitary symmetric group on a countable set. Let B be a countable basis of W_2 and observe that the finitary symmetric group on B is in fact a subgroup of $G_{2,1}$. This proves the case $m = 1$. It is also not difficult to see that H_2 is isomorphic to the semidirect product of the finitary symmetric group on $\mathbb{Z}$ with the infinite cyclic group generated by the 'shift' $s : z \longmapsto z+1$, $z \in \mathbb{Z}$. Now let B be the basis $\{a_1^i a_2, a_2^i a_1 \mid i \in \mathbb{N}\}$ whose lexicographic order is order-isomorphic to that of $\mathbb{Z}$ in its natural order, and let α be the element defined by the tree diagram in Fig. 10. Then α induces the shift s. Let $K \subset G_{2,1}$ be the finitary symmetric group on B, and put $H = \langle K, \alpha \rangle$. Then K is normal in H, and it is clear that restricting the action of H to B induces a homomorphism from H onto H_2. Suppose $k\alpha^t \in H$ lies in the kernel of this homomorphism, where $k \in K$. Since k affects only finitely many elements of B, we must have $t = 0$, which in turn implies $k = 1$. The case $m = 2$ is now complete.

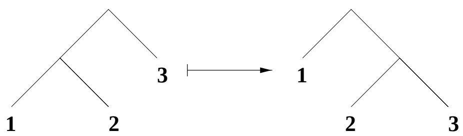
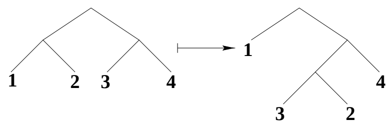


FIGURE 10

Now let $m \geq 3$, and define

$$\begin{aligned} \phi : \quad & S \longrightarrow W_2 \\ & (i,j) \longmapsto a_2^{j-1} a_1^i a_2 \,. \end{aligned}$$

Observe that ϕ is injective and, moreover, $S^\phi \cup \{a_2^m\}$ is a basis (see Fig. 11). Let $g_1 \in G_{2,1}$ be the element with tree diagram



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and, for $2 \leq i \leq m-1$, define $g_i \in G_{2,1}$ to be the element whose only non-trivial column is

$$\begin{pmatrix} a_2^{i-1} \\ g_1 \\ a_2^{i-1} \end{pmatrix}.$$

It is straightforward to check that $\phi g_i \phi^{-1} = h_i$ for $1 \leq i \leq m-1$. Since all g_i fix a_2^m and ϕ is a bijection between S and S^ϕ, we get that $G = \langle g_1, \ldots, g_{m-1} \rangle$ is a subgroup of $G_{2,1}$ isomorphic to H_m (observe that an element element of G is trivial if and only if it fixes S^ϕ pointwise). The proof is complete.

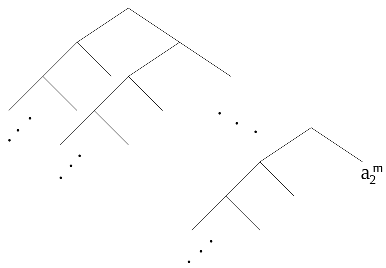


FIGURE 11

2.4 RESTRICTIONS ON SUBGROUPS

This section is about groups that cannot be embedded in any group $G_{n,r}$, and it therefore motivates and justifies the search for more finitely presented simple groups. The first serious restriction on subgroups of the groups $G_{n,r}$ was encountered by Higman [24]. Let us recall that the rank of an abelian group is the minimal integer r such that every finitely generated subgroup is generated by r elements.

***Theorem* 2.7** *Let $n \geq 2$ and $r \geq 1$ be integers. Let $A \subset G_{n,r}$ be torsion-free abelian group of finite rank. Then A is free and its centraliser in $G_{n,r}$ has finite index in its normaliser in $G_{n,r}$. Consequently, $G_{n,r}$ has no subgroup isomorphic to $SL_3(\mathbb{Z})$.*

Another, rather strong, restriction on torsion-free subgroups is implicit in the next theorem which was also proved by Higman [24]. Recall that a root of an element g of a group G is an element $h \in G$ such that $h^t = g$ for some $t \in \mathbb{N}$.

***Theorem* 2.8** *An element g of $G_{n,r}$ of infinite order does not have arbitrarily large roots, i.e., there is a bound on the t for which there may exist $h \in G_{n,r}$ with $h^t = g$.*

As a consequence of this the additive group of the rationals is not embeddable in any $G_{n,r}$. Also certain Baumslag-Solitar groups $BS(m,l)$ with $l \neq m$ are excluded as subgroups by this theorem. The group $BS(l,m)$ is generated by two elements a and b subject to the single defining relation $b^{-1}a^l b = a^m$. Suppose that $l = mk$ for some integer k. Then, if we write $a_i = a^{b^i}$, $i \in \mathbb{Z}$ we have $a_i^l = a_{i-1}^m$ and thus

$$a_0^m = a_1^{mk} = a_2^{mk^2} = (a_i^m)^{k^i},$$

which implies that a^m has arbitrarily large roots.

The last result can be contrasted with the following observation.

***Lemma* 2.9** *In $G_{n,r}$, every element of finite order has n^i-th roots for all $i \geq 0$.*

Proof. It certainly suffices to show that every element g of finite order has an n-th root. It is shown in Proposition 4.5 that g has a symbol Δ with $\operatorname{top}(\Delta) = \operatorname{bot}(\Delta)$, so that g induces a permutation of $\operatorname{top}(\Delta)$. Each cycle of this permutation corresponds to n cycles in $\operatorname{top}(\Gamma)$ when Γ is the symbol obtained from Δ by simply expanding all columns. So there is an element h in the symmetric group on $\operatorname{top}(\Gamma)$ whose n-th power induces exactly the same cycles in $\operatorname{top}(\Gamma)$ as g, i.e., $h^n = g$, as required.

In Section 4.2 we establish another restrictive result (Theorem 4.8). It is about torsion subgroups and gives as a special case the following.

***Theorem* 2.10** *Every torsion subgroup of $G_{n,r}$ is locally finite.*

Together with Theorem 2.4 this gives a complete characterisation of torsion subgroups of $G_{n,r}$; they are precisely the countable locally finite groups.

2.5 PING-PONG AND SOME FREE PRODUCTS

In [37] Scott gives an example of a non-abelian free subgroup of $G_{2,1}$ due to Higman. In this section we prove a probably well known Ping-Pong Lemma and give some examples of subgroups of $G_{n,r}$ to which it applies.

***Lemma* 2.11** (Ping-Pong Lemma) *Let G be a group acting faithfully on a set X by permutations. Let H and K be subgroups of order at least three of G, and assume X_H and X_K are disjoint non-empty subsets of X satisfying the following condition.*

(PP) *If h and k are non-trivial elements of H respectively K, then $X_K^h \subset X_H$ and $X_H^k \subset X_K$.*

Then the subgroup of G generated by H and K is isomorphic to the free product of H with K.

Proof. By the normal form theorem for free products (see for example [28]) we only have to show that no word of the form $h_1k_1 \cdots h_nk_n$, defines the trivial element, where $1 \neq h_i \in H$, $1 \neq k_i \in K$, with the only possible exceptions being h_1 and k_n but not both if $n = 1$. First suppose that $h_1 \neq 1$ and $k_n = 1$. Then condition (PP) implies that $x^{h_1k_1\cdots h_n} \in X_H$ for all $x \in X_K$, whence $h_1k_1 \cdots h_n$ cannot be trivial. A similar argument (this time with $x \in X_H$) applies if $h_1 = 1$ and $k_n \neq 1$. If, on the other hand, h_1 and k_n are both non-trivial, then we can conjugate by some $h \in H$ with $h \neq h_1$ ($|H| > 2$) to get to the first case. The case $h_1 = k_n = 1$ is reduced to the first case again by conjugation, and the lemma is proved.

***Corollary* 2.12** *The group $G_{n,r}$ has non-abelian free subgroups of every countable rank for every $n \geq 2$ and $r \geq 1$.*

Proof. We restrict ourselves to $G_{2,1}$, the general case being similar. Since the free group of rank two contains free groups of every countable rank, it suffices to find two elements $a, b \in G_{2,1}$ which freely generate a free subgroup. Define a and b by the tree diagrams in Fig. 12. To have a permutation action of $G_{2,1}$ we consider its induced action on the set of ends X of the infinite rooted binary tree, i.e., all infinite sequences with values in $\{a_1, a_2\}$. Then we apply the Ping-Pong Lemma with $G = G_{2,1}$, $H = \langle a \rangle$, $K = \langle b \rangle$, $X_H = a_2X$, and $X_K = a_1X$.

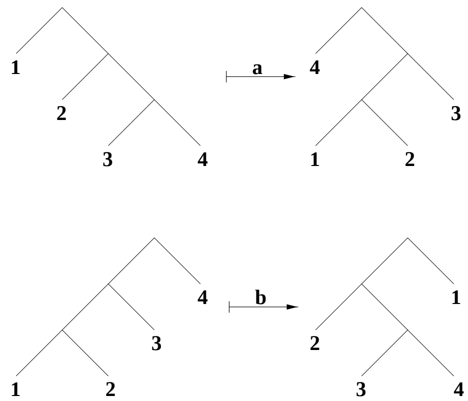


FIGURE 12

Remark. Observe that a and b are both cyclically order preserving so that, in fact, $T_{2,1}$ has free subgroups. This should be compared with Corollary 4.9 of [10] saying that $F_{n,r}$ has no non-abelian free subgroups. Note also that these generators are a little easier than those in [37].

***Corollary* 2.13** *Every free product of finitely many finite groups is embeddable in $G_{n,r}$ for any choice of $n \geq 2$ and $r \geq 1$.*

Proof. We show that for every finite group F, the free product of F with the infinite cyclic group generated by the element a defined in Fig. 12 is embeddable in $G_{2,1}$. This suffices, for we can take F to be the direct product of the given finite groups, $F_1, \ldots, F_t$ say, and it then follows from the normal form theorem for free products that the subgroup $\langle F_i^{a^i} \mid 1 \leq i \leq t \rangle$ of $F * \langle a \rangle$ is isomorphic to $F_1 * \cdots * F_t$. Let s be the order of F and let B be the basis $\{a_1^i a_2 \mid 0 \leq i \leq s-2\} \cup \{a_1^{s-1}\}$. Choose a bijection between B and F so that a_2 corresponds to the identity and consider the group K of permutations of B which is induced by the right regular representation of F on itself and this bijection. Now the Ping-Pong Lemma applies with $G = G_{2,1}$, $H = \langle a \rangle$, K, and X, X_H and X_K as in the proof of Corollary 2.12.

Remark. This result follows in a more indirect way from the fact that every free product of finitely many finite groups has a free subgroup of finite index (see for example [14]) together with Corollary 2.12 and Lemma 2.5

***Corollary* 2.14** *The free product of an infinite cyclic group with the direct product of finitely many finite and finitely generated free abelian groups is embeddable in $G_{n,r}$ for any choice of $n \geq 2$ and $r \geq 1$.*

Proof. Let $H_1, \ldots, H_r$ be a finite family of finitely generated free abelian groups and finite groups of order at least three. Let $K_i = \langle x_i \rangle$ be infinite cyclic groups for $1 \le i \le r$. Suppose that all these groups act on the same set X and that for $1 \le i \le r$ there are subsets X_{H_i}, X_{K_i} of X such that condition (PP) holds for every pair (H_i, K_i). Assume in addition that for every i, $1 \le i \le r$, H_i and K_i both fix $\bigcup_{j \ne i}(X_{H_j} \cup X_{K_j})$ pointwise. Then we claim that the group $M \subset \mathrm{Sym}(X)$ generated by the H_i and $x_1 x_2 \cdots x_r$ is isomorphic to $(H_1 \times \cdots \times H_r) * \mathbb{Z}$. It is clear from the assumptions that $[h_j, h_i] = [h_j, k_i] = [k_j, k_i] = 1$ for all $h_l \in H_l, k_l \in K_l$ and $1 \le i \ne j \le r$. So the group H generated by the H_i is isomorphic to $\times_{i=1}^{r} H_i$. Put $k = x_1 x_2 \cdots x_r$ and $K = \langle k \rangle$ and let $w = h_1 k_1 \cdots h_t k_t$ be a non-trivial word with $h_i \in H, k_i \in K$. We have to show that w is not the identity. If $t = 1$ and either h_1 or k_1 are trivial this is obvious. Otherwise w contains some non-trivial h_i which we may assume is h_1 after possibly conjugating w. Now h_1 can be assumed to be a word $l_1 \cdots l_s$ with $l_j \in H_j$ and clearly $l_q \ne 1$ for some q. Conjuagating w again we may now assume that $k_t = 1$ and h_n contains some $l'_q \in H_q$. Now consider $X_{K_q}^{w}$. By the hypothesis, only elements of H_q and k must be taken into account to see that $X_{K_q}^{w} \subset X_{H_q}$. Finally, to see that such a construction can be done inside $G_{n,r}$ use Corollary 2.12 and 2.13 and put such constructions further down in the tree to create space in order to meet our conditions above on fixed points. Note that the assumption on the orders of the finite groups is no problem, since we can consider the group we are looking for as a subgroup of a group satisfying these conditions. This completes the proof.

2.6 Generators for $G_{n,r}$

For future reference we include here some remarks on generating sets for the groups $G_{n,r}$ and $G_{n,r}'$. It was shown by Mason [30] that $G_{n,r}'$ can be generated by two elements. He assumed that $1 < r \le n$ which is no restriction, as $G_{n,r}$ is isomorphic to $G_{n,r+n-1}$ (see Section 2.3). The elements needed are all defined in Appendix B which can be folded out to be visible while reading here. This Appendix is also used in Section 4.5, and the dashed lines in the tree diagrams play a role then, but should be ignored for the time being. Mason proved that ab and c generate $G_{n,r}'$. We show that $G_{n,r}$ can also be generated by two elements. By Lemma 2.1, only for odd n is there something to prove. Observe, that cbc^{-1} is a single cycle, y say, in the symmetric group S on $\{x_j a_i \mid 1 \le j \le r, 1 \le i \le n\}$ fixing at most $x_1 a_1$. So y and $\hat{a}$ generate S which includes a, and hence $\langle a, b, c\rangle \subset \langle \hat{a}, b, c\rangle$. Since $\hat{a}$ and b commute and have coprime finite orders, $\langle \hat{a}, b\rangle = \langle \hat{a}b\rangle$. (This is also true for a in place of $\hat{a}$.) Hence, $\langle \hat{a}b, c\rangle = G_{n,r}$, because $\hat{a}$ is not even, if n is odd.

Chapter 3

Expansibility and Fake Expansibility

This chapter lays the ground for the constructions in Chapter 4. In the first half we record without proofs the main results of [35]. Those are the definition of E-expansible groups, a presentation for E-expansible groups and the θ-construction. In the second part we introduce fake-E-expansibility as a generalisation of E-expansibility and then we prove that certain fake-E-expansible groups are finitely presented if E is finitely presented.

3.1 Expansibility

In [35] Scott introduced the notion of E-expansibility in order to show that certain subgroups of $\mathcal{G}_{n,1}$ are finitely presented. Let us state an equivalent definition of E-expansibility.

Let E and H be subgroups of $\mathcal{G}_{n,1}$ with $E \subset H$. Then H is called E-*expansible* if there exists a generating system Y for H such that, if h is in H and $h = y_1 \cdots y_m$ with $y_i \in Y^{\pm 1}$ for $1 \leq i \leq m$, then there are E-symbols $\Gamma_1, \dots, \Gamma_m$ for $y_1, \dots, y_m$, respectively, such that the combination $\Gamma_1 \cdots \Gamma_m$ exists. Thanks to Lemma 1.3, this combination is an E-symbol for h, in particular, every element of H has an H-symbol. The idea behind this definition is that all relations which hold in H can be expressed in terms of E-symbols, and hence H can be investigated by studying E-symbols.

The main result of Scott [35] gives an explicit description of a set of defining relations for an E-expansible group H, when H contains $G_{n,1}$. As we are going to use it in the following chapter, we recall now the main features of that result whose actual statement comes only in the following section.

As in [35] we denote subgroups of $\mathcal{G}_{n,1}$ which contain $G_{n,1}$ by capital script letters. We consider the following situation: H is a subgroup of $\mathcal{G}_{n,1}$, $\mathcal{H} = \langle G_{n,1}, H \rangle$ is the subgroup of $\mathcal{G}_{n,1}$ generated by $G_{n,1}$ and H.

For $g \in \mathcal{G}_{n,1}$ define σ_g to be the element with symbol

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ g & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}.$$

It is easy to see that $H^* = \{\sigma_h \,|\, h \in H\}$ is a subgroup of $\mathcal{G}_{n,1}$ isomorphic to H (Lemma 7 in [35]). Let us also recall the following lemma which is Lemma 8 in [35] for which we indicate a tree-diagram proof. We hope that this is more enlightening than the original proof.

***Lemma* 3.1** *If $\mathcal{H}$ is H-expansible, then $\langle G_{n,1}, H^* \rangle$ and $\mathcal{H}$ are equal.*

Sketch of proof. Let $h \in H$. Thanks to the H-expansibility of $\mathcal{H}$, h has arbitrarily large H-symbols, for αh can be expressed as a combination of H-symbols for all $\alpha \in G_{n,1}$. The example in Fig. 13 shows how to prove $h \in \langle G_{n,1}, H^* \rangle$. Note that $a \in G_{n,1}$ and that the element in the second row of that figure is an element of $\langle G_{n,1}, H^* \rangle$.

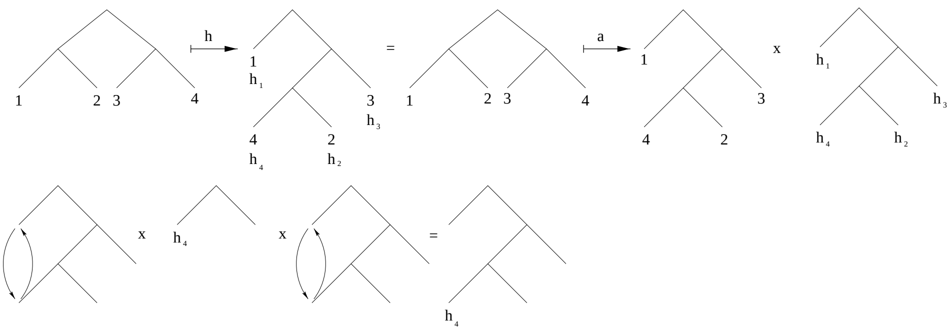


FIGURE 13

Since h_3 is an inescapable isomorphism, using the argument above, it has an H-symbol with at least one trivial middle row entry. Assume, for example, h_3 has the tree-diagram in Fig. 14.

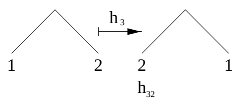


FIGURE 14

Then h has the tree-diagram in Fig. 15.

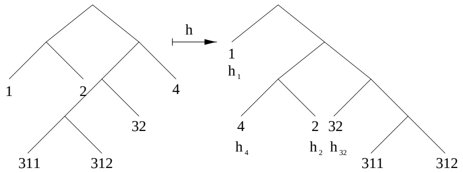


FIGURE 15

Now, let g and k be the elements defined in Fig. 16 and check that $ghk = \sigma_h$. So $H^* \subset \langle G_{n,1}, H \rangle$ and our sketch of the proof is complete.

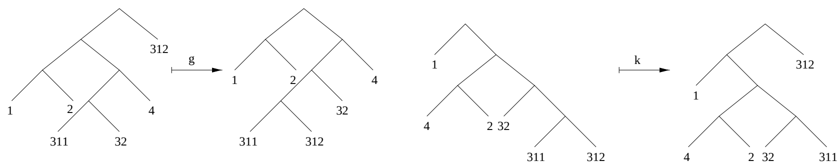


FIGURE 16

***Corollary* 3.2** *If $\mathcal{H}$ is H-expansible, then the following hold.*

(i) Every H-symbol defines an element of $\mathcal{H}$.

(ii) The class of subgroups of $\mathcal{H}$ is closed under restricted direct products of countable families and finite extensions.

Proof. As can be seen from the proof of the lemma, every H-symbol is the combination of finitely many conjugates of H-symbols for elements of H^* with a symbol for an element of $G_{n,r}$. This proves (i). Furthermore, from the proof of the lemma together with (i), we get that for every subgroup U of $\mathcal{H}$, U^* is a subgroup of $\mathcal{H}$. So given a countable family $U_1, U_2, \ldots$ of subgroups of $\mathcal{H}$, we can "place them below the vertices of an infinite basis" as indicated in Fig. 17 below. It is clear that this gives a subgroup of $\mathcal{G}_{n,1}$ isomorphic to the direct product of the U_i. Since the elements of the restricted direct product have finite supports, they have H-symbols, and the first part of (ii) is proved.

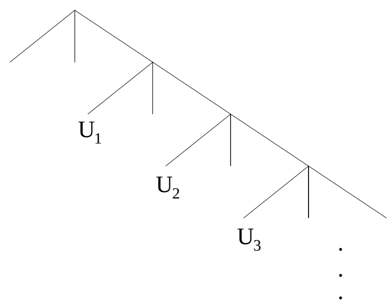


FIGURE 17

The second part of (ii) follows from the well known embedding theorem of Krasner and Kalužnin (see for example [33]), stating that every extension of a group G by a group H is isomorphic to a subgroup of the standard wreath product of G with H, cf. Section 1.1. Since the direct and restricted direct product of a finite family are indistinguishable, the corollary follows from the first part of (ii) and the fact that the symmetric group of any finite basis is a subgroup of $G_{n,1}$.

3.2 A PRESENTATION FOR EXPANSIBLE GROUPS

In this section we complete the description of the main result of [35], stated here as Theorem 3.3. A set $\{\eta_2, \ldots, \eta_s\}$ of elements of $G_{n,1}$ is said to be *of type s* if there is a basis $B = \{a_1, w_2, \ldots, w_s\}$ such that η_i naturally corresponds to the involution (a_1, w_i) in $\mathrm{Sym}B$, the symmetric group on the set B.

We now define four sets A, B, C, and D of relations. The relations in B, C, and D are written in terms of general elements of $G_{n,1}$ and H^*, not just on elements of a chosen generating set. The elements are considered as 'shorthand' notation for the words on the chosen generating set to which they correspond. Since we have relations A, it does not matter which of the words a particular element of $G_{n,1}$ or H^* is taken to represent. Usually a generating set $X \cup Y$ will be used, where X and Y are finite generating sets for $G_{n,1}$, respectively H^*.

Relation-set A**.** This consists of a fixed set of defining relations of $G_{n,1}$ and a fixed set of defining relations of H^* with respect to the chosen generating sets.

Relation-set B**.** Let F denote the subgroup of $G_{n,1}$ that fixes all words of the form $a_1 w$, $w \in W_n$. Then $B = \{\alpha\sigma_h = \sigma_h\alpha \,|\, h \in H, \alpha \in F\}$.

Relation-set C**.** Let δ be the element with symbol

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n a_1 & a_n a_2 & \cdots & a_n a_n \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ a_n a_1 & a_2 & \cdots & a_{n-1} & a_1 & a_n a_2 & \cdots & a_n a_n \end{pmatrix}.$$

Then $C = \{\delta\sigma_h\delta\sigma_{h'} = \sigma_{h'}\delta\sigma_h\delta \,|\, h, h' \in H\}$.

Relation-set D**.** Let $h \in H$ and let

$$\Gamma = \begin{pmatrix} u_1 & \cdots & u_s \\ h_1 & \cdots & h_s \\ v_1 & \cdots & v_s \end{pmatrix}$$

be an H-symbol for σ_h. Let $S = \{\eta_2, \ldots, \eta_s\}$ be a set of type s with corresponding basis $\{a_1, w_2, \ldots, w_s\}$ and define τ and ϵ by the symbols

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_s \\ 1 & 1 & \cdots & 1 \\ a_1 & w_2 & \cdots & w_s \end{pmatrix} \text{ and } \begin{pmatrix} a_1 & w_2 & \cdots & w_s \\ 1 & 1 & \cdots & 1 \\ v_1 & v_2 & \cdots & v_s \end{pmatrix},$$

respectively. Furthermore, let $R_{h,\Gamma,S}$ denote the relation

$$\sigma_h = \tau\sigma_{h_1}\eta_2\sigma_{h_2}\cdots\eta_s\sigma_{h_s}\eta_s\cdots\eta_2\epsilon. \tag{3.1}$$

Then D is the set of relations $R_{h,\Gamma,S}$ for all possible choices of subscripts.

Define the set χ of relations to be $A \cup B \cup C \cup D$.

***Theorem* 3.3** (Scott) *Let H be a subgroup of $\mathcal{G}_{n,1}$. If $\mathcal{H} = \langle G_{n,1}, H\rangle$ is H-expansible, then the set χ of relations is a set of defining relations for $\mathcal{H}$. Moreover, if H is finitely presented, then $A \cup B \cup C$ is finitely based, i.e., there is a finite subset of $A \cup B \cup C$ which implies all the relations in $A \cup B \cup C$.*

Before we recall a construction due to Scott for obtaining H-expansible groups in the next section, we give an example showing how restrictive expansibility is. Observe that H must be H-expansible if $\mathcal{H}$ is H-expansible.

Example. Define H to be the group of order two generated by the element h defined by the H-symbol

$$\begin{pmatrix} a_1^2 & a_1a_2 & a_2 \\ h & 1 & 1 \\ a_1^2 & a_2 & a_1a_2 \end{pmatrix}.$$

Then H is clearly H-expansible, but $\mathcal{H}$ is not, for $h\sigma_h$ has no H-symbol. This is because every H-symbol for h has a column

$$\begin{pmatrix} a_1^{2k} \\ h \\ a_1^{2k} \end{pmatrix},$$

whereas every H-symbol for σ_h has a column

$$\begin{pmatrix} a_1^{2k+1} \\ h \\ a_1^{2k+1} \end{pmatrix},$$

for $k \geq 0$. So $h\sigma_h$ cannot have an H-symbol which is the combination of H-symbols for h respectively σ_h. Observe that σ_h is indeed an element of $\mathcal{H}$ even though Lemma 3.1 does not apply. What is even worse, we cannot define an element h_1 such that h has the column

$$\begin{pmatrix} a_1^{2k+1} \\ h_1 \\ a_1^{2k+1} \end{pmatrix}$$

for some $k \geq 0$.

3.3 THE θ-CONSTRUCTION

Here we describe a method due to Scott to obtain expansible groups, cf. [35]. In Section 4.2 we prove a general theorem about groups obtained in this fashion. Recall from Section 1.1 the definition of the permutational wreath product. It is clear that for groups A, B, and a permutation group C, each homomorphism $\varphi : A \longrightarrow B$ induces an obvious homomorphism $A \wr C \longrightarrow B \wr C$.

Now consider the following situation. Let F be a finitely generated free group and denote by $\mathcal{S}_n$ the symmetric group of degree n with its natural action on $\{1, 2, \ldots, n\}$. Let $F \wr \mathcal{S}_n$ be the permutational wreath product, where F acts on itself via the right regular representation. Suppose we are given a homomorphism $\theta : F \longrightarrow F \wr \mathcal{S}_n$. Then θ induces a homomorphism $\theta_1 : F \wr \mathcal{S}_n \longrightarrow F \wr \mathcal{S}_n \wr \mathcal{S}_n$, and iterating this process gives the following commutative diagram of homomorphisms

$$\begin{array}{ccccccccc}
F & \xrightarrow{\theta} & F \wr \mathcal{S}_n & \xrightarrow{\theta_1} & F \wr \mathcal{S}_n \wr \mathcal{S}_n & \xrightarrow{\theta_2} & F \wr \mathcal{S}_n \wr \mathcal{S}_n \wr \mathcal{S}_n & \xrightarrow{\theta_3} & \\
 & & \downarrow \rho_1 & & \downarrow \rho_2 & & \downarrow \rho_1 & & \cdots \\
 & & \mathcal{S}_n & \xleftarrow{\eta_1} & \mathcal{S}_n \wr \mathcal{S}_n & \xleftarrow{\eta_2} & \mathcal{S}_n \wr \mathcal{S}_n \wr \mathcal{S}_n & \xleftarrow{\eta_3} &
\end{array}$$

where ρ_i and η_i denote the natural projections. Define $\psi_1 = \theta\rho_1$, $\psi_i = \theta\theta_1 \cdots \theta_{i-1}\rho_i$ for $i \geq 2$ and let K_i be the kernel of ψ_i. Finally put $K = \bigcap_{i=1}^{\infty} K_i$ and $H_\theta = F/K$. The following proposition summarises Lemmas 14, 15, and 17, and Theorem 2 of [35].

***Proposition* 3.4** *Let H_θ be defined as above. Then there is an embedding of H_θ in $\mathcal{G}_{n+1,1}$ whose image is denoted by H_θ again such that, the group $\mathcal{H}_\theta = \langle G_{n+1,1}, H_\theta \rangle$ is H_θ-expansible and every element of H_θ has a H_θ-symbol of the form*

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n & a_{n+1} \\ h_1 & h_2 & \cdots & h_n & 1 \\ a_{1\pi} & a_{2\pi} & \cdots & a_{n\pi} & a_{n+1} \end{pmatrix}, \tag{3.2}$$

where $\pi \in \mathcal{S}_n$. Moreover, $\mathcal{H}_\theta$ is finitely presented whenever H_θ is finitely presented.

Observe that the special form (3.2) of the H_θ-symbols for elements of H_θ implies that every (simple) expansion of any H_θ-symbol exists, and is again an H_θ-symbol. This immediately gives the following result which we state for future reference.

***Lemma* 3.5** *Let Δ and Γ be H_θ-symbols for some $k \in \mathcal{H}_\theta$. Suppose that Δ is an expansion of Γ and let B and C be finite bases in W_n so that $B \succeq \mathrm{top}(\Delta)$ and $C \succeq \mathrm{bot}(\Delta)$. Then the following hold.*

(i) $\mathrm{top}(\Delta) \succeq \mathrm{top}(\Gamma)$ *and* $\mathrm{bot}(\Delta) \succeq \mathrm{bot}(\Gamma)$.

(ii) There are unique H_θ-symbols Δ_1 and Δ_2 for k such that $\mathrm{top}(\Delta_1) = B$ *and* $\mathrm{bot}(\Delta_2) = C$. *Furthermore, Δ_1 and Δ_2 can be constructed effectively and $\Delta \preceq \Delta_1$ as well as $\Delta \preceq \Delta_2$.*

3.4 Fake Symbols and Fake Expansibility

The following definitions are inspired by the results of the last section which roughly say that an expansible group in which the elements have sufficiently well expansible symbols is a good candidate for being finitely presented.

Let S_n be the subgroup of $G_{n,1}$ consisting of the elements which have a symbol of the form

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 1 & 1 & \cdots & 1 \\ a_{1\pi} & a_{2\pi} & \cdots & a_{n\pi} \end{pmatrix} \tag{3.3}$$

where π is some permutation of the set $\{1, \ldots, n\}$. Obviously, S_n is isomorphic to the symmetric group of degree n. Let E and H be subgroups of $\mathcal{G}_{n,1}$ with $E \subset H$. Then a *fake-E-symbol for $h \in H$* is a symbol for h of the form

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_s \\ x_1 & x_2 & \cdots & x_s \\ v_1 & v_2 & \cdots & v_s \end{pmatrix}$$

where $x_i \in E \cup S_n$ for $1 \le i \le s$. Furthermore we call H *fake-E-expansible* if there exists a set of generators Y for H such that, if h is in H and $h = y_1 \cdots y_m$ with $y_i \in Y^{\pm 1}$ for $1 \le i \le m$, then h has a symbol which is the combination $\Gamma_1 \cdots \Gamma_m$ of fake-E-symbols $\Gamma_1, \ldots, \Gamma_m$ for $y_1, \ldots, y_m$, respectively. Note that this combination is rather a $\langle E, S_n \rangle$-symbol than a fake-E-symbol. The next two results relate fake expansibility and expansibility.

***Proposition* 3.6** *Let E and H be subgroups of $\mathcal{G}_{n,1}$ and $E \subset H$. If H is fake-E-expansible and every $e \in E$ has a fake-E-symbol of the form*

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ x_1 & x_2 & \cdots & x_n \\ a_{1\pi} & a_{2\pi} & \cdots & a_{n\pi} \end{pmatrix} \tag{3.4}$$

for some permutation π (possibly depending on e) of the set $\{1, \ldots, n\}$, then H is E-expansible.

Proof. We must show that there is a generating system Y for H such that, if $y_1 \cdots y_m$ is any word in $Y^{\pm 1}$, then there are E-symbols $\Gamma_1, \ldots, \Gamma_m$ for $y_1, \ldots, y_m$, respectively, such that their combination $\Gamma_1 \cdots \Gamma_m$ exists. To this end let Y be a generating system for H such that H is fake-E-expansible with respect to Y. Let $y_1 \cdots y_m$ be any word in $Y^{\pm 1}$ and choose fake-E-symbols $\Gamma_1, \ldots, \Gamma_m$ for $y_1, \ldots, y_m$, respectively, such that the combination $\Gamma_1 \cdots \Gamma_m$ exists. Observe that all of the Γ_k have the same number of columns, C say, and that we can assume (by reordering if necessary) that the bottom row of Γ_k is the same as the top row of Γ_{k+1} as ordered sets, for $1 \leq k \leq m-1$. Now enumerate the columns of each Γ_j, $1 \leq j \leq m$, from left to right by $1, \ldots, C$ and apply the following procedure.

Start with Step 1 and stop when you reach Step $m+1$ where for $1 \leq i \in \mathbb{N}$ Step i is defined as follows.

Step *i*: Let X be the set of all numbers of columns whose middle row entry in Γ_i is a non-trivial element of S_n.

(a) If X is empty then go to Step $i+1$.

(b) If X is not empty do the following.
For $1 \leq j \leq m$ simply-expand all the columns of Γ_j numbered by some $x \in X$ and denote the resulting symbol by Γ_j again. Now reorder the columns in $\Gamma_1, \ldots, \Gamma_m$ so that the bottom row of Γ_k is equal to the top row of Γ_{k+1} as ordered sets, for $1 \leq k \leq m-1$. Replace C by $C + |X|(n-1)$ and go to Step 1.

Note that (3.3) and (3.4) ensure we can execute part (b) and that C will again be the number of columns of each of the Γ_i. To prove that the algorithm gives the required result we will show, by induction on $i \leq m+1$, that

(α) Step i will be reached and

(β) at the time of reaching Step i all of the symbols $\Gamma_1, \ldots, \Gamma_{i-1}$ are E-symbols.

Part (β) holds because the only way to reach Step i is via Step $i-1$ part (a), $i \geq 2$.

As Step 1 is reached by definition, to prove (α) it suffices to show that Step i will finally lead to Step $i+1$, $1 \leq i \leq m$. First let $i = 1$. As part (a) of Step 1 leads immediately to Step 2, we only need to check part (b). By the definition of X and (3.3) it is clear that at the end of Step 1 part (b) Γ_1 is an E-symbol, and the procedure continues with Step 1 part (a) which leads to Step 2.

Now consider i with $m \geq i > 1$. Again Step i part (a) leads to Step $i+1$. If, on the other hand, we have to execute Step i part (b), then the definition of X and (3.3) show that, at the end of this, Γ_i is an E-symbol. Furthermore, using (β), if $1 \leq c \leq C$ and for some j with $1 \leq j \leq i-1$ the c^{th} column of Γ_j has now a non-trivial middle row entry in S_n, then the c^{th} column of Γ_i has trivial middle row entry. Thus, while going from Step 1 up to Step i again, only columns with trivial middle row entry in Γ_i are affected. Hence, using the induction hypothesis, we will come to Step i part (a) which leads to Step $i+1$. This completes the proof of the proposition.

Next we show that with a little more care we can even deduce the E-expansibility of $\mathcal{H} = \langle H, G_{n,1} \rangle$.

***Proposition* 3.7** *Let E and H be subgroups of $\mathcal{G}_{n,1}$ with $E \subset H$ and let H be fake-E-expansible. Assume that there is a generating system Y for H such that $E \subset Y$ and every $y \in Y$ has a fake-E-symbol of the form*

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ x_1 & x_2 & \cdots & x_n \\ a_{1\pi} & a_{2\pi} & \cdots & a_{n\pi} \end{pmatrix}$$

for some permutation π of the set $\{1, \ldots, n\}$. Then $\mathcal{H} = \langle H, G_{n,1} \rangle$, the group generated by H and $G_{n,1}$, is E-expansible.

Proof. In the light of Proposition 3.6, it suffices to prove that $\mathcal{H}$ is fake-E-expansible with respect to the generating system $Z = Y \cup G_{n,1}$. Let $h = z_1 \cdots z_m$, $z_i \in Z^{\pm 1}$, $1 \leq i \leq m$. The proof is by induction on m. If $m = 1$ it follows from the assumptions that $h = z_1$ has a fake-E-symbol. So we assume that there are fake-E-symbols $\Gamma_1, \ldots, \Gamma_{m-1}$ for $z_1, \ldots, z_{m-1}$, respectively, such that the combination $\Gamma_1 \cdots \Gamma_{m-1}$ exists.

First let $z_m \in Y$. Then z_m has a fake-E-symbol Γ of the form (3.4), and as $E \subset Y$, every middle row entry of Γ has such a fake-E-symbol too. An induction on the number of simple expansions needed to go from $\{a_1, \ldots, a_n\}$ to $\mathrm{bot}(\Gamma_{m-1})$ now shows that we can expand Γ to a fake-E-symbol Γ_m for z_m such that the combination $\Gamma_1 \cdots \Gamma_m$ exists.

In the case $z_m \in G_{n,1}$, let Γ be a 1-symbol for z_m. Then there is a common expansion $\mathcal{U}$ of the two finite bases $\mathrm{top}(\Gamma)$ and $\mathrm{bot}(\Gamma_{m-1})$. The argument in the proof of Proposition 3.6 also shows that we can expand all the Γ_k, $1 \leq k \leq m-1$, to obtain fake-E-symbols Δ_k for z_k such that the combination $\Delta_1 \cdots \Delta_{m-1}$ exists and $\mathrm{bot}(\Delta_{m-1}) = \mathcal{U}$. But there is also an expansion Δ_m of Γ with $\mathrm{top}(\Delta_m) = \mathcal{U}$. Hence, the combination $\Delta_1 \cdots \Delta_m$ exists and therefore $\mathcal{H}$ is fake-E-expansible, as required.

3.5 PRESENTATIONS FOR FAKE EXPANSIBLE GROUPS

This section is entirely devoted to the proof of the following theorem.

***Theorem* 3.8** *Let E, Y, H, and $\mathcal{H}$ satisfy the hypotheses of Proposition 3.7. Then $\mathcal{H}$ is finitely presented, if E is finitely presented.*

To begin with, we show that under the hypotheses of the theorem, $\mathcal{H}$ is in fact generated by $G_{n,1}$ and E. Write $\mathcal{E} = \langle G_{n,1}, E\rangle$ and note that $\mathcal{E}$ is E-expansible as a subgroup of the E-expansible group $\mathcal{H}$, cf. Proposition 3.7. So $\mathcal{E}$ is the group of all those elements of $\mathcal{G}_{n,1}$ that have an E-symbol, by Corollary 3.2. Since all elements of Y (or H if you like) have an E-symbol, $\mathcal{H} = \mathcal{E}$.

Now let Z be a finite generating set for E. All what follows will be with respect to the generating set $Z^* \cup G_{n,1}$ for $\mathcal{H}$. This is a generating set by Lemma 3.1. By Theorem 3.3, $\chi = A \cup B \cup C \cup D$ is a set of defining relations for $\mathcal{H}$, where A, B, C, and D are the sets of relations defined in Section 3.2. We define three more sets of relations as follows.

Relation-set D''. Let $e \in E$ and let

$$\Gamma = \begin{pmatrix} u_1 & \cdots & u_s \\ x_1 & \cdots & x_s \\ v_1 & \cdots & v_s \end{pmatrix}$$

be a fake-E-symbol for σ_e. Let $S = \{\eta_2, \ldots, \eta_s\}$ be a set of type s with corresponding basis $\{a_1, w_2, \ldots, w_s\}$ and define τ and ϵ by the symbols

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_s \\ 1 & 1 & \cdots & 1 \\ a_1 & w_2 & \cdots & w_s \end{pmatrix} \text{ and } \begin{pmatrix} a_1 & w_2 & \cdots & w_s \\ 1 & 1 & \cdots & 1 \\ v_1 & v_2 & \cdots & v_s \end{pmatrix},$$

respectively. Furthermore, let $R''_{e,\Gamma,S}$ denote the relation

$$\sigma_e = \tau\sigma_{x_1}\eta_2\sigma_{x_2}\cdots\eta_s\sigma_{x_s}\eta_s\cdots\eta_2\epsilon.$$

Then D'' is the set of relations $R''_{e,\Gamma,S}$ for all possible choices of subscripts.

Note that this differs from the definition of the set D only in the use of 'fake-E-symbol' in place of 'E-symbol'. But clearly each E-symbol for σ_e is also a fake-E-symbol for σ_e, so that $D \subset D''$.

Relation-sets D' and $\hat{D}$. Let $e \in E$ and let

$$\Gamma = \begin{pmatrix} a_1 & \cdots & a_n \\ x_1 & \cdots & x_n \\ a_{1\pi} & \cdots & a_{n\pi} \end{pmatrix}$$

be the fake-E-symbol (3.4) which exists by hypothesis. Let δ be the same element as in the definition of the relation set C and let δ_i, $1 \leq i \leq n$ act on the basis

$$\{a_1, a_2, \dots, a_{n-1}, a_n a_1 a_1, \dots, a_n a_1 a_n, a_n a_2, \dots, a_n a_n\}$$

as the involution that interchanges a_1 with $a_n a_1 a_i$ and fixes all the other elements. We let R'_e denote the relation

$$\sigma_e = \delta\delta_1\sigma_{x_1}\delta_2\sigma_{x_2}\cdots\delta_n\sigma_{x_n}\delta_n\cdots\delta_1\delta\sigma_\pi. \tag{3.5}$$

Then $D' = \{R'_e \,|\, e \in E\}$ and $\hat{D} = \{R'_e \,|\, e \in Z\}$.

Remark. As the elements σ_e, δ, δ_i ($1 \leq i \leq n$) play a significant role throughout the remainder of this section, we have provided reminding tree diagrams in Appendix C which can be folded out.

Observe that $\hat{D}$ is finite, and our first aim is to show that together with $A \cup B \cup C$ it implies D' (Lemma 3.11). Define $\chi' = A \cup B \cup C \cup D'$ and $\chi'' = A \cup B \cup C \cup D''$. As $A \cup B \cup C$ is finitely based, by Theorem 3.3, the theorem will then follow immediately from the next lemma using $\chi \subset \chi''$.

***Lemma* 3.9** *Every relation in χ' is a consequence of relations in χ and each relation in χ'' is a consequence of relations in χ'.*

Before we prove Lemma 3.9 we need a few more facts. The next result is Lemma 12 in [35].

***Lemma* 3.10** *If $\{\eta_2, \dots, \eta_s\}$ is of type s and μ is a permutation of the set $\{m, \dots, r\}$, $1 \leq m \leq r \leq s$, then the relation*

$$\sigma_{g_m}\eta_{m+1}\sigma_{g_{m+1}}\cdots\eta_r\sigma_{g_r} = \alpha_\mu\sigma_{g_{m\mu}}\eta_{m+1}\sigma_{g_{(m+1)\mu}}\cdots\eta_r\sigma_{g_{r\mu}}\beta_\mu$$

is a consequence of $A \cup B \cup C$, where α_μ and β_μ have symbols

$$\begin{pmatrix} w_2 & \cdots & w_m & x_m & x_{m+1} & \cdots & x_r & w_{r+1} & \cdots & w_s \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ w_2 & \cdots & w_m & a_1 & w_{m+1} & \cdots & w_r & w_{r+1} & \cdots & w_s \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} w_2 & \cdots & w_m & w_{m+1} & \cdots & w_r & a_1 & w_{r+1} & \cdots & w_s \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ w_2 & \cdots & w_m & y_{m+1} & \cdots & y_r & y_{r+1} & w_{r+1} & \cdots & w_s \end{pmatrix}$$

$(x_{m\mu^{-1}} = a_1,\ x_j = w_{j\mu}\ (j\mu \neq m),\ y_{i+1} = w_{i\mu+1}\ (i\mu \neq r))$.

Let us record a few relations implied by $A \cup B \cup C$ which will be used soon. But first observe that $\{\delta_1, \ldots, \delta_n\}$ is a subset of a set of type $3n-2$. Let $S = \{\eta_2, \ldots, \eta_s\}$ be any set of type s and $e, e' \in E$, then the following hold.

$$\eta_i^2 = 1 \qquad \text{for } 2 \le i \le s \tag{3.6}$$

$$\eta_i\eta_j\sigma_e\eta_j = \eta_j\sigma_e\eta_j\eta_i \qquad \text{for } 2 \le i \ne j \le s \tag{3.7}$$

$$\sigma_e\eta_i\sigma_{e'}\eta_i = \eta_i\sigma_{e'}\eta_i\sigma_e \qquad \text{for } 2 \le i \le s \tag{3.8}$$

The definition of a set of type s gives immediately (3.6). Equations (3.7) follow from B and the fact that $\eta_j\eta_i\eta_j$ fixes a_1 if $i \ne j$. Finally, if we assume that $\{a_1, w_2, \ldots, w_s\}$ is the basis corresponding to S and define τ_i, $2 \le i \le s$, to be the element of $G_{n,1}$ given by a symbol whose only non-trivial columns are

$$\begin{pmatrix} a_n a_1 \\ 1 \\ w_i \end{pmatrix} \text{ and } \begin{pmatrix} w_i \\ 1 \\ a_n a_1 \end{pmatrix},$$

then τ_i fixes a_1 and $\tau_i\eta_i\tau_i = \delta$. Hence (3.8) is a conjugate of a relation in C under τ_i.

***Lemma* 3.11** *Let $e \in E$ have the fake-E-symbol*

$$\begin{pmatrix} a_1 & \cdots & a_n \\ y_1 & \cdots & y_n \\ a_{1\rho} & \cdots & a_{n\rho} \end{pmatrix},$$

then the relation $R'_e \in D'$ is a consequence of $\hat{\chi} = A \cup B \cup C \cup \hat{D}$.

Proof. We use induction on the length of e with respect to the generating set Z. If e has length one, then R'_e is already in $\hat{D}$, so we assume that $e = fz$ with $z \in Z$, $f \in E$ and f is of strictly shorter length than e. Suppose f and z have the fake-E-symbols

$$\begin{pmatrix} a_1 & \cdots & a_n \\ f_1 & \cdots & f_n \\ a_{1\alpha} & \cdots & a_{n\alpha} \end{pmatrix} \text{ and } \begin{pmatrix} a_1 & \cdots & a_n \\ z_1 & \cdots & z_n \\ a_{1\beta} & \cdots & a_{n\beta} \end{pmatrix},$$

respectively. Since $\sigma_e = \sigma_f\sigma_z$ is a consequence of A,

$$\sigma_e = \delta\delta_1\sigma_{f_1}\delta_2\sigma_{f_2}\cdots\delta_n\sigma_{f_n}\delta_n\cdots\delta_1\delta\sigma_\alpha\delta\delta_1\sigma_{z_1}\delta_2\sigma_{z_2}\cdots\delta_n\sigma_{z_n}\delta_n\cdots\delta_1\delta\sigma_\beta$$

is a consequence of $\hat{\chi}$. It is easy to see that $\delta\sigma_\alpha\delta$ fixes a_1 and hence commutes with every σ_{z_i} for $1 \le i \le n$, by B. That $\delta\sigma_\alpha\delta\delta_j = \delta_{j\alpha^{-1}}\delta\sigma_\alpha\delta$ holds in $G_{n,1}$ is also easy to check, whence

$$\sigma_e = \delta\delta_1\sigma_{f_1}\cdots\delta_n\sigma_{f_n}\delta_n\cdots\delta_1\delta_{1\alpha^{-1}}\sigma_{z_1}\cdots\delta_{n\alpha^{-1}}\sigma_{z_n}\delta_{n\alpha^{-1}}\cdots\delta_{1\alpha^{-1}}\delta\sigma_\alpha\sigma_\beta$$

is a consequence of $\hat{\chi}$. Clearly, $\sigma_\alpha\sigma_\beta = \sigma_{\alpha\beta}$, and using (3.7) and (3.8) we get

$$\sigma_e = \delta\delta_1\sigma_{f_1}\sigma_{z_{1\alpha}}\delta_2\sigma_{f_2}\sigma_{z_{2\alpha}}\cdots\delta_n\sigma_{f_n}\sigma_{z_{n\alpha}}\delta_n\cdots\delta_1\delta\sigma_{\alpha\beta}$$

as a consequence of $\hat{\chi}$. But

$$\begin{pmatrix} a_1 & \cdots & a_n \\ f_1 z_{1\alpha} & \cdots & f_n z_{n\alpha} \\ a_{1\alpha\beta} & \cdots & a_{n\alpha\beta} \end{pmatrix}$$

is the combination of the two given symbols for f and z and, by Lemma 1.4, $\rho = \alpha\beta$ and $y_i = f_i z_{i\alpha}$. Furthermore, $\sigma_\rho = \sigma_\alpha\sigma_\beta$ is a consequence of A and $\sigma_{y_i} = \sigma_{f_i}\sigma_{z_{i\alpha}}$ is a consequence of A whenever both, f_i and $z_{i\alpha}$, are either in E or S_n. But this always holds, for we know that y_i is in $E \cup S_n$. Thus R'_e is a consequence of $\hat{D}$, as required.

***Lemma* 3.12** *Every fake-E-symbol for σ_e is an expansion of*

$$\Gamma = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ e & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}.$$

Proof. Let Σ be the fake-E-symbol

$$\begin{pmatrix} u_1 & \cdots & u_s \\ x_1 & \cdots & x_s \\ v_1 & \cdots & v_s \end{pmatrix}$$

for σ_e. Then $\{u_1, \ldots, u_2\}$ is a finite basis and therefore an expansion of $\{a_1, \ldots, a_n\}$. As any expansion of $\{a_1, \ldots, a_n\}$ leads to an expansion Δ of Γ such that Δ is a fake-E-symbol for σ_e, there is a fake-E-symbol Δ for σ_e of the form

$$\begin{pmatrix} u_1 & \cdots & u_s \\ y_1 & \cdots & y_s \\ z_1 & \cdots & z_s \end{pmatrix}$$

which is an expansion of Γ. It follows, by Lemma 1.4, that $x_i = y_i$ and $z_i = v_i$ for $1 \le i \le s$, which implies $\Delta = \Sigma$. Thus the lemma is proved.

Proof of Lemma 3.9. Let us first show that every relation in χ' follows from the relations in χ. To this end let $e \in E$ have the fake-E-symbol (3.4) and let J be the subset of $\{1, 2, \ldots, n\}$ satisfying $x_j \in S_n$ if and only if $j \in J$. Let $K = \{1, 2, \ldots, n\} \backslash L$ and suppose $J = \{j_1, \ldots, j_r\}$ and $K = \{k_1, \ldots, k_t\}$. Then

$$\Gamma = \begin{pmatrix} a_n a_1 & \cdots & a_n a_n & a_2 & \cdots & a_{n-1} & a_1 a_{k_1} & \cdots & a_1 a_{k_t} \\ 1 & \cdots & 1 & 1 & \cdots & 1 & x_{k_1} & \cdots & x_{k_t} \\ a_n a_1 & \cdots & a_n a_n & a_2 & \cdots & a_{n-1} & a_1 a_{k_1\pi} & \cdots & a_1 a_{k_t\pi} \end{pmatrix}$$

$$\left.\begin{matrix} a_1a_{j_1}a_1 & \cdots & a_1a_{j_1}a_n & \cdots & a_1a_{j_r}a_1 & \cdots & a_1a_{j_r}a_n \\ 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 \\ a_1a_{j_1}\pi a_{1x_{j_1}} & \cdots & a_1a_{j_1}\pi a_{nx_{j_1}} & \cdots & a_1a_{j_r}\pi a_{1x_{j_r}} & \cdots & a_1a_{j_r}\pi a_{nx_{j_r}} \end{matrix}\right)$$

is an E-symbol for σ_e. Note that Γ has $s = rn + t + 2n - 2$ columns, and let $S = \{\eta_2, \ldots, \eta_s\}$ be the set of type s corresponding to the ordered basis

$$\{a_1, a_na_2, \ldots, a_na_n, a_2, \ldots, a_{n-1}, a_na_1a_{k_1}, \ldots, a_na_1a_{k_t},$$

$$a_na_1a_{j_1}a_1, \ldots, a_na_1a_{j_1}a_n, \ldots, a_na_1a_{j_s}a_1, \ldots, a_na_1a_{j_s}a_n\}.$$

As all this might be a little confusing, Appendix C shows schematic pictures of the top row of Γ and this basis with the appropriate labelling used in the definition of the relation set D. We consider the relation $R_{e,\Gamma,S}$ in χ (see (3.1)); namely

$$\sigma_e = \tau\eta_2\cdots\eta_{2n-2}\eta_{2n-1}\sigma_{x_{k_1}}\cdots\eta_{2n-2+t}\sigma_{x_{k_t}}\eta_{2n-2+t}\cdots\eta_2\epsilon$$

after the obvious cancellations using (3.6). Note that $\tau = \delta$ and $\eta_{2n-2+i} = \delta_{k_i}$. So, by using (3.6), (3.7), and (3.8), we obtain

$$\sigma_e = \delta\sigma_{x_{k_1}}^{\delta_{k_1}}\cdots\sigma_{x_{k_t}}^{\delta_{k_t}}\epsilon$$

as a consequence of χ. The relation $\epsilon = \sigma_{x_{j_1}}^{\delta_{j_1}}\cdots\sigma_{x_{j_r}}^{\delta_{j_r}}\delta\sigma_\pi$ is a consequence of A (see Appendix A). Thus

$$\sigma_e = \delta\sigma_{x_{k_1}}^{\delta_{k_1}}\cdots\sigma_{x_{k_t}}^{\delta_{k_t}}\sigma_{x_{j_1}}^{\delta_{j_1}}\cdots\sigma_{x_{j_r}}^{\delta_{j_r}}\delta\sigma_\pi$$

is a consequence of χ, and, using (3.7) and (3.8) again, we get R'_e (see (3.5)) as a consequence of χ, as required.

We now turn to the the second part of the lemma, whose proof is similar to that of Lemma 19 in [35]: we show that every relation in D'' is a consequence of χ'. Let

$$\Delta = \begin{pmatrix} u_1 & \cdots & u_s \\ x_1 & \cdots & x_s \\ v_1 & \cdots & v_s \end{pmatrix}$$

be a fake-E-symbol for σ_e, $e \in E$. Then, by Lemma 3.12, Δ is an expansion of the symbol in Lemma 3.12, and by Lemma 3.10 we can, and will, assume that the columns of Δ are in any order which suits us. The proof is by induction on the number m of columns of the form

$$\begin{pmatrix} a_1u \\ l \\ a_1v \end{pmatrix}$$

in Δ. The pattern of the induction is as follows. We first prove the cases $m = 1$ and $m = n$ and then turn to the induction step.

If $m = 1$, then

$$\Delta = \begin{pmatrix} a_1 & u_2 & \cdots & u_s \\ e & 1 & \cdots & 1 \\ a_1 & u_2 & \cdots & u_s \end{pmatrix}$$

and the corresponding relation in χ'' is $\sigma_e = \tau\sigma_e\epsilon$. But $\tau\epsilon = 1$ is a consequence of A, in this case, and τ fixes a_1, so the result is a consequence of $A \cup B$.

If $m = n$, then

$$\Delta = \begin{pmatrix} a_1a_1 & \cdots & a_1a_n & u_{n+1} & \cdots & u_s \\ x_1 & \cdots & x_n & 1 & \cdots & 1 \\ a_1a_{1\pi} & \cdots & a_1a_{n\pi} & u_{n+1} & \cdots & u_s \end{pmatrix},$$

by (3.4), and the corresponding relation in $D'' \subset \chi''$ is

$$\sigma_e = \tau\sigma_{x_1}\eta_2\sigma_{x_2}\cdots\eta_n\sigma_{x_n}\eta_n\cdots\eta_2\epsilon. \tag{3.9}$$

Let α be the element with symbol

$$\begin{pmatrix} a_1 & w_2 & \cdots & w_n & w_{n+1} & \cdots & w_{s-1} & w_s \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_na_1a_2 & \cdots & a_na_1a_n & z_{n+1} & \cdots & z_{s-1} & a_na_1a_1 \end{pmatrix},$$

for some $z_{n+1}, \ldots, z_{s-1}$. The relations $\alpha^{-1}\eta_i\alpha = \delta_i$, $2 \le i \le n$ are consequences of A and α fixes a_1, so

$$\sigma_e = \delta\delta_1\alpha^{-1}\sigma_{x_1}\eta_2\sigma_{x_2}\cdots\eta_n\sigma_{x_n}\eta_n\cdots\eta_2\alpha\delta_1\delta\sigma_\pi$$

is obtained from R'_e (see (3.5)) as a consequence of χ'. The element $\tau\alpha\delta_1\delta$ fixes a_1, because

$$\begin{array}{lllllllll} a_1a_i & \overset{\tau}{\mapsto} & w_i & \overset{\alpha}{\mapsto} & a_na_1a_i & \overset{\delta_1}{\mapsto} & a_na_1a_i & \overset{\delta}{\mapsto} & a_1a_i, \text{ if } 2 \le i \le n \\ a_1a_1 & \mapsto & a_1 & \mapsto & a_1 & \mapsto & a_na_1a_1 & \mapsto & a_1a_1 \end{array}.$$

Hence, using B and (3.6),

$$\sigma_e = \tau\alpha\delta_1\delta\delta\delta_1\alpha^{-1}\sigma_{x_1}\eta_2\cdots\eta_n\sigma_{x_n}\eta_n\cdots\eta_2\alpha\delta_1\delta\sigma_\pi\delta\delta_1\alpha^{-1}\tau^{-1}$$

is a consequence of χ'. Furthermore, the relation $\alpha\delta_1\delta\sigma_\pi\delta\delta_1\alpha^{-1}\tau^{-1} = \epsilon$ follows from A (see Appendix A), whence (3.9) is a consequence of χ'.

We now turn to the inductive step. Consider the simple expansion

$$\begin{pmatrix} u_1 & \cdots & u_{s-1} & u_sa_1 & \cdots & u_sa_n \\ x_1 & \cdots & x_{s-1} & y_1 & \cdots & y_n \\ v_1 & \cdots & v_{s-1} & v_s(a_1\pi) & \cdots & v_s(a_n\pi) \end{pmatrix}$$

of Δ. For a given set of type $r = s + n - 1$ we have to show that the relation

$$\sigma_e = \tau\sigma_{x_1}\eta_2\sigma_{x_2}\cdots\eta_{s-1}\sigma_{x_{s-1}}\eta_s\sigma_{y_1}\cdots\eta_r\sigma_{y_n}\eta_r\cdots\eta_2\epsilon$$

follows from χ'. By the inductive hypothesis we can assume that, for any set $\{\nu_2, \ldots, \nu_s\}$ of type s,

$$\sigma_e = \tau'\sigma_{x_1}\nu_2\sigma_{x_2}\cdots\nu_s\sigma_{x_s}\nu_s\cdots\nu_2\epsilon' \tag{3.10}$$

is a consequence of χ'. Let β be the element with symbol

$$\begin{pmatrix} a_1 & w_2 & \cdots & w_{s-1} & w_s & w_{s+1} & \cdots & w_r \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ a_1 & z_2 & \cdots & z_{s-1} & z_s a_1 & z_s a_2 & \cdots & z_s a_n \end{pmatrix}$$

for some $z_2, \ldots, z_s$ and define $\nu_i = \beta^{-1}\eta_i\beta$, $2 \leq i \leq s-1$. Then there exists ν_s such that $\{\nu_2, \ldots, \nu_s\}$ is of type s with corresponding basis $\{a_1, z_2, \ldots, z_s\}$. With the symbol

$$\begin{pmatrix} z_2 & \cdots & z_s & a_1 a_1 & \cdots & a_1 a_n \\ 1 & \cdots & 1 & y_1 & \cdots & y_n \\ z_2 & \cdots & z_s & a_1 a_{1\pi} & \cdots & a_1 a_{n\pi} \end{pmatrix}$$

for σ_{x_s}, the case $m = n$ implies that

$$\sigma_{x_s} = \tau''\eta_2\cdots\eta_s\sigma_{y_1}\eta_{s+1}\cdots\sigma_{y_n}\eta_r\cdots\eta_2\epsilon'' \tag{3.11}$$

is a consequence of χ'. The relations $\tau''\eta_2\cdots\eta_{s-1} = \nu_s\beta^{-1}$, $\epsilon''\nu_s\cdots\nu_2\epsilon' = \epsilon$ and $\tau'\beta^{-1} = \tau$ are consequences of A (see Appendix A), and hence

$$\begin{aligned} \sigma_e &= \tau'\sigma_{x_1}\nu_2\sigma_{x_2}\cdots\nu_s\sigma_{x_s}\nu_s\cdots\nu_2\epsilon' \\ &= \tau'\beta^{-1}\sigma_{x_1}\eta_2\sigma_{x_2}\cdots\eta_{s-1}\sigma_{x_{s-1}}\beta\nu_s\sigma_{x_s}\nu_s\cdots\nu_2\epsilon' \\ &= \tau'\beta^{-1}\sigma_{x_1}\eta_2\cdots\sigma_{x_{s-1}}\beta\nu_s\tau''\eta_2\cdots\eta_s\sigma_{y_1}\eta_{s+1}\cdots\sigma_{y_n}\eta_r\cdots\eta_2\epsilon''\nu_s\cdots\nu_2\epsilon' \\ &= \tau\sigma_{x_1}\eta_2\cdots\sigma_{x_{s-1}}\eta_s\sigma_{y_1}\eta_{s+1}\cdots\sigma_{y_n}\eta_r\cdots\eta_2\epsilon \end{aligned}$$

is a consequence of χ'. Here the equalities follow from (3.10), the definition of the ν_i and B as β fixes a_1, (3.11), and the argument of the preceding paragraph respectively. This completes the proof of Lemma 3.9 and Theorem 3.8.

Chapter 4

Subgroups: Part II

In this chapter we will describe finitely presented simple groups which have more complicated subgroups than the groups $G_{n,r}$. In the first section we state without proofs the results of Scott showing that $GL_n(\mathbb{Z})$ can be embedding in finitely presented simple groups for all $n \geq 1$. Recall Theorem 2.7; $GL_3(\mathbb{Z})$ is not a subgroup of any of the groups $G_{n,r}$. In Section 4.2 we prove that finitely generated infinite torsion groups cannot be contained in any of the finitely presented simple groups with solvable conjugacy problem constructed so far. The following section introduces a well known class of finitely generated infinite torsion groups. These are the Grigorchuk-Gupta-Sidki groups. In the remaining two sections of the chapter we construct finitely presented simple groups that have subgroups isomorphic to Grigorchuk-Gupta-Sidki groups.

4.1 Some Expansible Groups

Using the θ-construction (see Section 3.3) Scott was able to prove the following two theorems [36].

***Theorem* 4.1** *For every $n \geq 1$ there is a finitely presented simple group containing $\mathbb{Z}^n \rtimes GL_n(\mathbb{Z})$ as a subgroup.*

In fact it is shown that $\mathbb{Z}^n \rtimes GL_n(\mathbb{Z})$ is isomorphic to H_θ for a suitable homomorphism $\theta : F \longrightarrow F \wr \mathrm{Sym}_{2^n-1}$, where F is a free group of rank $2n+1$

To understand the next result, let us recall some facts about abelian groups. Let A be a countable abelian group and let T be the set of all periodic elements. Then T is a characteristic subgroup of A, and A/T is called the torsion factor group of A.

***Theorem* 4.2** *Let m be an integer. Every countable abelian group whose torsion factor group is a finitely generated $\mathbb{Z}[\frac{1}{m}]$-module is embeddable in a finitely presented simple group.*

This theorem is proved by showing that for $m \in \mathbb{N}$ the group with presentation $\langle a, b \mid b^{-1}ab = a^m \rangle$ is isomorphic to H_θ for a homomorphism $\theta : F \longrightarrow F \wr \mathrm{Sym}_n$, where F is free group of rank 2 and suitable $n \in \mathbb{N}$. This result should be compared to Theorem 2.8.

For completeness, let us also record the following theorem, which was obtained by Scott [37] with quite a different kind of expansible groups. It uses a construction of C. F. Miller III [31] for finitely presented groups with an unsolvable conjugacy problem which are extensions of one finitely generated free group by another free group of finite rank. A group G is said to have an unsolvable conjugacy problem if there is no algorithm which decides whether two arbitrary given elements of G are conjugate in G.

***Theorem* 4.3** *There exist finitely presented simple groups with unsolvable conjugacy problem.*

4.2 Torsion Local Finiteness

The main result of this section shows that all the groups $G_{n,r}$ and the groups constructed by Scott in [36], i.e., those groups of Theorems 4.1 and 4.2, are torsion locally finite (Theorem 4.8). A group is called *torsion locally finite (t.l.f.)* if every torsion subgroup is locally finite, or equivalently, if every finitely generated torsion subgroup is finite. We believe this result to be of independent interest. Our main interest, though, comes from the fact that it shows that the groups considered in Sections 4.5 and 4.6 are genuinely new finitely presented simple groups. Let us start with an observation.

***Lemma* 4.4** *The class of torsion locally finite groups is closed under extensions and subgroups.*

Proof. Only the statement about extensions requires proving. To this end, let N be a normal subgroup of the group G, let $\pi : G \longrightarrow G/N$ be the natural projection and assume N and G/N are both t.l.f. Let S be a finitely generated torsion subgroup of G. We have to show that S is finite. Clearly, S^π is a finitely generated torsion subgroup of G/N and therefore finite. Let K be the kernel of the restriction of π to S, i.e., $K = N \cap S$. Then $|S : K|$ is finite, whence K is a finitely generated torsion subgroup of N and hence finite. So S is finite, as an extension of the finite group K by the finite group S^π. The lemma is proved.

Recall the θ-construction from Section 3.3, in particular, Proposition 3.4, which implies that H_θ acts length preserving on W_n. Recall also Lemma 3.5. We move on to examine the structure of finite subgroups of $\mathcal{H}_\theta$. Suppose K is a subgroup of $\mathcal{H}_\theta$. A basis B is said to be a K*-basis* if it is finite and every element of K has an H_θ-symbol whose top and bottom row are equal to B.

***Proposition* 4.5** *If K is a finite subgroup of H_θ, then there is a K-basis.*

Proof. It suffices to find a finite basis B such that, for every $b \in B$, each $k \in K$ has a column

$$\begin{pmatrix} b \\ h \\ c \end{pmatrix}$$

with $c \in B$ and $h \in H_\theta$. To this end let $K = \{k_1, \ldots, k_t\}$ ($t = |K|$) and for $1 \le i \le t$ let Γ_i be an H_θ-symbol for k_i. Let C be a finite basis of W_n contained in $\bigcap_{i=1}^t \operatorname{top}(\Gamma_i) W_n$. In particular $C \succeq \operatorname{top}(\Gamma_i)$ for $1 \le i \le t$, and hence there are (unique) H_θ-symbols Δ_i for k_i with $\operatorname{top}(\Delta_i) = C$, by Lemma 3.5(*ii*). Let $U = \bigcap_{i=1}^t \operatorname{bot}(\Delta_i) W_n$ and $B = B_U$, i.e. $B = \bigsqcup_{i=1}^t \operatorname{bot}(\Delta_i)$. We claim that B has the required properties.

First let $u \in U$. Then $u \in \operatorname{bot}(\Delta_i) W_n$ for $1 \le i \le t$ and, by Lemma 3.5 (*i*), there are $v_i \in CW_n$ and $h_i \in H_\theta$ such that k_i has the column

$$\begin{pmatrix} v_i \\ h_i \\ u \end{pmatrix}.$$

Choose $k \in K$ and note that $B \succeq C$, since $1 \in K$. By Lemma 3.5, k has a column

$$\begin{pmatrix} u \\ h \\ w \end{pmatrix}$$

with (unique) $w \in W_n$ and $h \in H_\theta$. Hence

$$\begin{pmatrix} v_i \\ h_i h \\ w \end{pmatrix}$$

is a column of $k_{j_i} = k_i k$, and $v_i \in CW_n$ implies $w \in \operatorname{bot}(\Delta_{j_i}) W_n$ for $1 \le j_i \le t$. Thus $w \in U$.

Now let $b \in B$, in particular, $b \in U$. Then, by Lemma 1.1 b), $b \in \operatorname{bot}(\Delta_j)$ for some j, $1 \le j \le t$. Hence

$$\begin{pmatrix} c \\ h \\ b \end{pmatrix}$$

is a column of k_j, where $c \in C$, $h \in H_\theta$. For $k \in K$ with column

$$\begin{pmatrix} b \\ h' \\ w \end{pmatrix}$$

we see that

$$\begin{pmatrix} c \\ hh' \\ w \end{pmatrix}$$

is a column of Δ_l where $k_l = k_j k$. Together with the previous paragraph it follows that $w \in U \cap \mathrm{bot}(\Delta_l)$. So, by Lemma 1.1 c), $w \in B$, as required.

Addendum. The above proof actually establishes a more general result. Namely, if D is a finite basis contained in $\bigcap_{i=1}^{t} \mathrm{top}(\Gamma_i) W_n$, then

$$\overline{D}^K = \bigsqcup_{i=1}^{t} \mathrm{bot}(\Sigma_i) \tag{4.1}$$

is a K-basis, where Σ_i is an H_θ-symbol for k_i with $\mathrm{top}(\Sigma_i) = D$. We call $\overline{D}^K$ *the K-closure of D.*

Let us now describe subgroups K of $\mathcal{H}_\theta$ that admit a K-basis.

***Lemma* 4.6** *Let $K \subset \mathcal{H}_\theta$ have a K-basis B. Then K embeds in the permutational wreath product $H_\theta \wr \mathrm{Sym}(B)$ of H_θ with the symmetric group on the set B, where H_θ acts on itself via right multiplication.*

Proof. Let $B = \{b_1, \ldots, b_s\}$. Then each $k \in K$ has an H_θ-symbol of the form

$$\begin{pmatrix} b_1 & b_2 & \ldots & b_s \\ h_1(k) & h_2(k) & \ldots & h_s(k) \\ b_1\pi_k & b_2\pi_k & \ldots & b_s\pi_k \end{pmatrix}$$

with $h_i(k) \in H_\theta$ and $\pi_k \in \mathrm{Sym}(B)$. We leave it to the reader to check that the map ϕ defined by

$$\begin{array}{rrcl} \phi: & K & \longrightarrow & H_\theta \wr \mathrm{Sym}(B) \\ & k & \longmapsto & (h_1(k), h_2(k), \ldots, h_s(k))\pi_k \end{array}$$

is an embedding.

***Theorem* 4.7** *Let H_θ be a group defined by a homomorphism $\theta : F \longrightarrow F \wr \mathrm{Sym}_{n-1}$ as in Section 3.3. If H_θ is torsion locally finite, then $\mathcal{H}_\theta = \langle G_{n,1}, H_\theta \rangle$ is also torsion locally finite.*

Proof. By Lemma 4.4, $H_\theta \wr \mathrm{Sym}(B)$ is t.l.f. for any finite set B. Thanks to the previous lemma, it therefore suffices to show that every finitely generated torsion subgroup K of $\mathcal{H}_\theta$ admits a K-basis. The proof is by induction on the minimal number, d, of generators for K, the case $d = 1$ follows from Proposition 4.5. So we assume that $d > 1$, and $K = \langle k_1, \dots, k_d \rangle$. Define $G = \langle k_1, \dots, k_{d-1} \rangle$ and $X = \langle k_d \rangle$. By the induction hypothesis, G and X are finite and have G- and X-bases A_0 and A_1, respectively. Let $A = A_0 \sqcup A_1$ and define inductively, for $i \geq 0$,

$$B_0 = \overline{A}^X, \;\; C_i = \overline{B_i}^G \text{ and } B_{i+1} = \overline{C_i}^X.$$

Note that the B_i and C_i are (finite) X- respectively G-bases. Furthermore, the proof of (4.1) shows that for each $b_i \in B_i$, $i \geq 1$, there are $c_i \in C_{i-1}$, $h_i \in H_\theta$, and $x_i \in X$ such that

$$\begin{pmatrix} c_i \\ h_i \\ b_i \end{pmatrix}$$

is a column of x_i. Similarly, for each $c_i \in C_{i-1}$, $i \geq 1$, there are $d_i \in B_{i-1}$, $l_i \in H_\theta$, and $g_i \in G$ such that

$$\begin{pmatrix} d_i \\ l_i \\ c_i \end{pmatrix}$$

is a column of g_i. Repetition of these arguments shows that for all $i \geq 0$ and $b \in B_i$ there are $\overline{b} \in B_0$, $h_b \in H_\theta$, and an element $k_b \in K = \langle G, X \rangle$ such that

$$\begin{pmatrix} \overline{b} \\ h_b \\ b \end{pmatrix}$$

is a column of k_b.

Assume now, in order to obtain a contradiction, that the chain

$$A \preceq B_0 \preceq C_0 \preceq B_1 \preceq C_1 \preceq \cdots \tag{4.2}$$

does not terminate. Then we can find a not eventually constant sequence $b_0, b_1, b_2, \dots$ of elements of W_n with $b_i \in B_i$ and $b_i \leq b_{i+1}$ for all $i \geq 0$. Since B_0 is finite there exist $i > j$ such that $b_i \neq b_j$ but $\overline{b_i} = \overline{b_j}$, showing that $k = k_{b_j}^{-1} k_{b_i}$ has the column

$$\begin{pmatrix} b_j \\ h_{b_j}^{-1} h_{b_i} \\ b_i \end{pmatrix}.$$

Using $b_i > b_j$ and the fact that H_θ acts in a length preserving manner, it is easy to prove that k has infinite order, which is a contradiction. Thus the chain (4.2) terminates and it is obvious that its last element is a K-basis. This completes the proof of the theorem.

Note that all the groups H_θ used in [36] are either torsion-free or isomorphic to $\mathbb{Z}^n \rtimes GL_n(\mathbb{Z})$ and therefore t.l.f. (see for example [33]). Using also the fact that $G_{n,r}$ embeds in $G_{n,1}$ (see Section 1.4), we have the following.

***Theorem* 4.8** *All the groups $G_{n,r}$, $G'_{n,r}$ and all the finitely presented simple groups constructed by Scott in [36], i.e., the groups of Theorems 4.1 and 4.2, are torsion locally finite.*

At this point we should also mention that the finitely presented simple groups recently constructed by Burger and Mozes [9] are t.l.f., indeed torsion free. This follows immediately from their presentation as a free product with amalgamation $F *_A F$, where F and A are finitely generated free groups. Therefore, Theorem 4.8 tells us that none of the previously known finitely presented simple groups can be isomorphic to any of the groups $\mathcal{H}_{f,p}{}'$ which will be constructed in the following sections, with one possible exception; the finitely presented simple groups with unsolvable conjugacy problem of [37] (Theorem 4.3).

4.3 Grigorchuk-Gupta-Sidki

In the following we give a short introduction to Grigorchuk-Gupta-Sidki groups (GGS groups hereafter). This terminology was introduced in [2] by Baumslag. We first describe the 'original' Grigorchuk groups as defined in [18]. Then we turn to closely related groups including those investigated in [22] by N. Gupta and Sidki. Originally, Grigorchuk defined his groups by their action on the unit interval, whereas Gupta and Sidki considered automorphisms of rooted regular trees. Later it became popular to describe all these groups as subgroup of the automorphism group of a rooted regular tree (see for example [2, 34]). This approach appears more intuitive.

We begin with the description of Grigorchuk groups. Our description here will be in the language of inescapable isomorphisms. But it is straightforward to extend these inescapable isomorphisms to graph automorphisms of the tree W_p described in Section 1.2 (see Section 5.3). Instead of n we now use p, since it makes the comparison of the original definition with ours a little easier. (In [18] p was always a prime, but for our purpose this is irrelevant.) Given an infinite sequence $\omega = \omega(0)\omega(1)\ldots$ with values in $\{0, 1, \ldots, p-1\}$, define the inescapable isomorphism b_ω of W_p by its action on the basis

$$\{{a_p}^r a_1 a_1, \ldots, {a_p}^r a_1 a_p, {a_p}^r a_2, \ldots, {a_p}^r a_{p-1} \,|\, r \geq 0\},$$

the action is

$$\begin{array}{lll} {a_p}^r a_1 a_i & \mapsto & {a_p}^r a_1 (a_i \pi^{\omega(r)}), \quad 1 \le i \le p \\ {a_p}^r a_j & \mapsto & {a_p}^r a_j, \qquad\qquad\quad 2 \le j \le p-1 \end{array}$$

where π denotes the cyclic permutation $(a_1\, a_2 \ldots a_p)$. See Fig. 18 for an example.

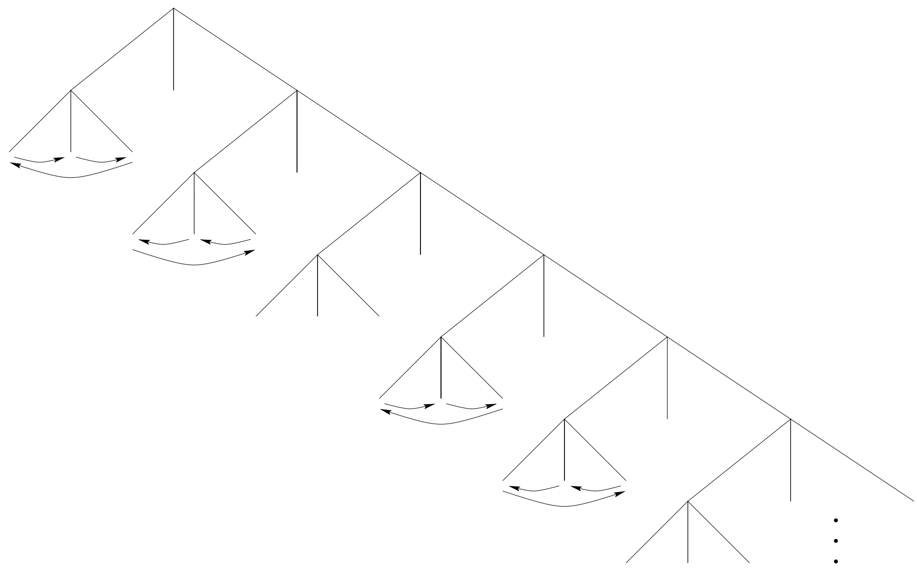


FIGURE 18: THE ELEMENT b_ω WITH $\omega = 120120120\cdots$ AND $p = 3$

Given a sequence ω with values in $\{0, 1, \ldots, p\}$ define the two maps $\bar{}$ and $\tilde{}$ from $\{0, \ldots, p\}$ to $\{0, \ldots, p-1\}$ as in [18], i.e.,

$$\bar{x} = \begin{cases} 0, & \text{if } x = p \\ 1, & \text{if } x \neq p \end{cases} \quad \text{and} \quad \tilde{x} = \begin{cases} 1, & \text{if } x = p \\ x, & \text{if } x \neq p \end{cases}.$$

The *Grigorchuk group* G_ω is isomorphic to the subgroup of $\mathcal{G}_{p,1}$ generated by $b_{\bar{\omega}}$, $b_{\tilde{\omega}}$ and the inescapable isomorphism γ with symbol

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_{p-1} & a_p \\ 1 & 1 & \ldots & 1 & 1 \\ a_2 & a_3 & \ldots & a_p & a_1 \end{pmatrix}. \tag{4.3}$$

As mentioned above, in [18] the groups G_ω are described by their action on the set $\mathcal{C}$ of all open intervals of the real line of the form $(\frac{t}{p^k}, \frac{t+1}{p^k})$ with $0 \le t \le p^k - 1$ and $1 \le k \in \mathbb{N}$. The isomorphism between G_ω as described above and G_ω as described in [18] is induced by the bijection between W_p

and the set $\mathcal{C}$ that lets $a_{i_1}a_{i_2}\cdots a_{i_k} \in W_p$ correspond to $(\frac{t}{p^k}, \frac{t+1}{p^k})$, where $t = \sum_{j=1}^{k} \frac{i_j - 1}{p^j}$.

Remark. It is worth noting that the nature of the maps $^-$ and $\tilde{}$ is irrelevant for our arguments. However, they played a crucial role in [18] for deducing that G_ω is a finitely generated infinite torsion group of intermediate growth in case each of the letters $0, 1, \ldots, p$ occurs infinitely many times in ω.

Let us quickly recall the definition of some other groups which were considered during the search for groups of intermediate growth in the early eighties. N. Gupta and S. Sidki investigated in [22] the group G_{GS} which is defined for $p = 3$ and generated by the element γ with symbol (4.3) and b which has the fake-$\langle b\rangle$-symbol

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ \gamma & \gamma^{-1} & b \\ a_1 & a_2 & a_3 \end{pmatrix}.$$

J. Fabrykowski and Gupta studied in [15] and [16] the group G_ω for $p = 3$ and $\omega = 111\cdots$ with the task to show that it has intermediate growth. Their proofs, however are not complete. Recently L. Bartholdi [1] has given a unified method to show that several two generated GGS-groups are of intermediate growth.

The most general sort of groups similar to those above are called special groups by Grigorchuk [20]. They are defined as follows. Let R be a subgroup of the symmetric group on $\{a_1, \ldots a_p\}$. Now let $\omega = \omega(0)\omega(1)\omega(2)\cdots$ be an infinite sequence where each $\omega(i)$ is a $(p-1)$-tupel of elements of R. Assume $\omega(i) = (\omega(i)_1, \ldots, \omega(i)_{p-1})$ and define b_ω to be the element with symbol

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{p-1} & a_p \\ \omega(i)_1 & \omega(i)_2 & \cdots & \omega(i)_{p-1} & b_{\omega^\kappa} \\ a_1 & a_2 & \cdots & a_{p-1} & a_p \end{pmatrix},$$

where κ is the forget map (4.4). A *special group* is a group generated by R and a finite set D of elements b_ω.

4.4 THE GROUPS $E_{f,p}$ AND $\mathcal{H}_{f,p}$

Here we describe some fake expansible groups which are used to prove that every Grigorchuk group G_ω which is defined by an almost periodic sequence ω can be embedded in a finitely presented simple group (Theorem 4.10). More general examples are given in Section 4.6. The reason for singling out the groups in this section is that our investigation of decision problems in Chapters 5 and 6 deals with these groups only.

Fix $p, f \in \mathbb{N}$ and define $\Omega = \{0, \ldots, p-1\}^{\mathbb{N}_0}$; thus, if $\omega \in \Omega$ then $\omega = \omega(0)\omega(1)\omega(2)\cdots$. For $n \in \mathbb{N}$ define the sequence $[n]$ by

$$[n](i) = \begin{cases} 1 & \text{if } i \equiv 0 \,(\mathrm{mod}\, n) \\ 0 & \text{otherwise} \end{cases}$$

The "forget" map κ is defined by

$$\begin{array}{cccc} \kappa \;: & \Omega & \longrightarrow & \Omega \\ & \omega = \omega(0)\omega(1)\ldots & \longmapsto & \omega^{\kappa} = \omega(1)\omega(2)\ldots \end{array} \tag{4.4}$$

Define $\Omega_0 = \{[pf]^{\kappa^i} \,|\, 0 \le i \le pf-1\}$ and let $E_{f,p}$ be the subgroup of $\mathcal{G}_{p,1}$ generated by the set $\{b_\omega \,|\, \omega \in \Omega_0\}$, where b_ω is defined as in Section 4.3. Observe that for $0 \le i \le pf-1$

$$[pf]^{\kappa^i}(j) = \begin{cases} 1 & \text{if } j \equiv -i (\mathrm{mod} pf) \\ 0 & \text{otherwise} \end{cases}$$

Note that if we define the sum of the two sequences ω, ω' by $(\omega + \omega')(i) = (\omega(i) + \omega'(i))(\mathrm{mod}\, p)$ then we have $b_\omega b_{\omega'} = b_{\omega+\omega'}$. And a glance at Fig. 18 tells us that $b_\omega{}^p = 1$ for all $\omega \in \Omega_0$. Furthermore, b_ω has the symbol

$$\left(\begin{array}{ccccc} a_1 & a_2 & \ldots & a_{p-1} & a_p \\ \gamma^{\omega(0)} & 1 & \ldots & 1 & b_{\omega^\kappa} \\ a_1 & a_2 & \ldots & a_{p-1} & a_p \end{array} \right), \tag{4.5}$$

where γ is the "top spin" with symbol (4.3)). Since $\gamma \in S_p$ and $[pf]^{\kappa^i} = [pf]^{\kappa^{pf+i}}$ for $i \in \mathbb{N}$, we have that $E_{f,p}$ is fake-$E_{f,p}$-expansible (with respect to the generating system $E_{f,p}$) and each element e of $E_{f,p}$ has a fake-$E_{f,p}$-symbol of the form

$$\left(\begin{array}{ccccc} a_1 & a_2 & \cdots & a_{p-1} & a_p \\ \gamma^k & 1 & \cdots & 1 & e' \\ a_1 & a_2 & \cdots & a_{p-1} & a_p \end{array} \right), \tag{4.6}$$

where $0 \le k \le p-1$, $e' \in E_{f,p}$.

Define $\mathcal{H}_{f,p}$ to be the subgroup of $\mathcal{G}_{p,1}$ generated by $G_{p,1}$ and $E_{f,p}$. Applying Proposition 3.7 with $H = Y = E = E_{f,p}$ we get the following lemma.

***Lemma* 4.9** *The group $E_{f,p}$ is isomorphic to the direct product of pf cyclic groups of order p, and the group $\mathcal{H}_{f,p} = \langle E_{f,p}, G_{p,1} \rangle$ is $E_{f,p}$-expansible.*

In the following section we show that the group $\mathcal{H}_{f,p}$ defined above has a finitely presented simple commutator subgroup which contains an isomorphic copy of each Grigorckuk group G_ω if ω is periodic of period f.

4.5 The Commutator Subgroup of $\mathcal{H}_{f,p}$

Let us call the infinite sequence ω *almost periodic* if there exist $k, f \in \mathbb{N}$, such that $\omega(i) = \omega(i+f)$ for $k \leq i \in \mathbb{N}$. In this case the least such f is called the *period* of ω. We also say that the sequence ω is *almost f-periodic.*

***Theorem* 4.10** *Let $\mathcal{H}_{f,p}$ be the group defined in the previous section. The derived subgroup of $\mathcal{H}_{f,p}$ is a finitely presented simple group and contains every Grigorchuk group G_ω which is defined by an almost f-periodic sequence ω.*

Proof. The simplicity of $\mathcal{H}_{f,p}{}'$ follows from Lemma 2.3. Since $E_{f,p}$ is finite, we can apply Theorem 3.8, to get that $\mathcal{H}_{f,p}$ is finitely presented. So $\mathcal{H}_{f,p}{}'$ will be finitely presented if it has finite index in $\mathcal{H}_{f,p}$ (Coroll 2.8 in [28]). Thanks to Lemmas 2.1 and 4.9, $x^{2p} \in \mathcal{H}_{f,p}{}'$ for all $x \in G_{p,1} \cup E_{f,p}$. As $G_{p,1} \cup E_{f,p}$ contains a finite generating system for $\mathcal{H}_{f,p}$, we get that $\mathcal{H}_{f,p}/\mathcal{H}_{f,p}{}'$ is a finitely generated abelian group of exponent at most $2p$ and hence finite, as required.

Now it only remains to show that $\mathcal{H}_{f,p}{}'$ contains the required Grigorchuk groups. First note that if ω is any fp-periodic sequence then $b_\omega \in E_{f,p}$. Thus $\mathcal{H}_{f,p}$ contains $\sigma_{b_{\hat{\omega}}}$ where $\hat{\omega}$ is fp-periodic and begins with

$$\underbrace{\underbrace{10\ldots0}_{f}\underbrace{20\ldots0}_{f}\ldots\underbrace{(p-1)0\ldots0}_{f}\underbrace{0\ldots0}_{f}}_{pf}.$$

Let α be the element with symbol

$$\begin{pmatrix} a_2a_1 & \cdots & a_2a_{p-1} & a_1 & a_2a_p & a_3 & \cdots & a_p \\ 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \\ a_1a_1 & \cdots & a_1a_{p-1} & a_1a_p & a_2 & a_3 & \cdots & a_p \end{pmatrix} \tag{4.7}$$

and observe that $\alpha^{-1}\sigma_{b_\omega}\alpha = \sigma_{b_{0\omega}}$ and hence $\alpha^{-f}\sigma_{b_{\hat{\omega}}}^{-1}\alpha^f\sigma_{b_{\hat{\omega}}} = \sigma_{b_{[f]}} \in \mathcal{H}_{f,p}{}'$. Furthermore, for every f-periodic sequence ω, $\sigma_{b_\omega} \in \mathcal{H}_{f,p}{}'$.

As $E^* \cong E$ for every subgroup $E \subset \mathcal{G}_{p,1}$ (see Section 3.2 for the definition of E^*), in order to complete the proof it suffices to exhibit σ_γ and $\sigma_{b_{v\overline{0}}}$ as elements of $\mathcal{H}_{f,p}{}'$ for every finite sequence v with entries in $\{0, 1, \ldots p-1\}$, where $\overline{0} = 000\ldots$ and γ has the symbol (4.3). By Lemma 2.1, $G'_{p,1} = G_{p,1}$ if p is even and $\sigma_\gamma \in \mathrm{Alt}(\{a_1a_1, \ldots, a_1a_p, a_2, \ldots, a_p\}) \subset G'_{p,1}$ if p is odd, where $\mathrm{Alt}(Y)$ denotes the alternating group on the set Y. Therefore, $\sigma_\gamma \in \mathcal{H}_{f,p}{}'$ and $\sigma_{\sigma_\gamma} \in \mathcal{H}_{f,p}{}'$. From $\sigma_{b_{1\overline{0}}} = \sigma_{\sigma_\gamma}$ and the last paragraph it follows that $\sigma_{b_{v\overline{0}}} \in \mathcal{H}_{f,p}{}'$ for every finite sequence v with entries in $\{0, 1, \ldots p-1\}$. The proof of Theorem 4.10 is complete.

Remark. It can be shown that $\mathcal{H}_{f,p}'$ contains in fact subgroups isomorphic to any Grigorchuk group defined by an almost fp-periodic sequence. For such a group is clearly a subgroup of $\langle\gamma, E_{f,p}\rangle$ which, in turn, is a finite extension of $\langle\gamma, E_{f,p}\rangle \cap \mathcal{H}_{f,p}'$, and therefore embeddable in $\mathcal{H}_{f,p}'$, by Corollary 3.2. This approach, however, does not establish the natural embedding that is obtained in the proof of the theorem.

***Corollary* 4.11** *The group $\mathcal{H}_{f,p}$ is generated by three elements.*

This follows from Section 2.6 and the fact that E^* is generated by $b_\omega^{\alpha^i}$, $0 \le i \le pf-1$, where $\omega = [pf]$ and α is the element with symbol (4.7). To be more precise, $\mathcal{H}_{f,p}$ is generated by $b_{[pf]}$ and the elements $\hat{a}b$ and c, with $\hat{a}$, b, and c as defined in Appendix B. By noting that c is always an element of $G_{n,r}'$, we see that the index of $\mathcal{H}_{f,p}'$ in $\mathcal{H}_{f,p}$ is at most εp because $(\hat{a}b)^\varepsilon, (b_{[pf]})^p \in \mathcal{H}_{f,p}'$, where ε is the largest common divisor of 2 and $p-1$. In fact, we have the following.

***Corollary* 4.12** *The group $\mathcal{H}_{f,p}/\mathcal{H}_{f,p}'$ is cyclic of order εp, where ε is the largest common divisor of* 2 *and $p-1$. In particular, $G_{p,1} \cap \mathcal{H}_{f,p}' = G_{p,1}'$.*

Proof. For an fp-periodic sequence ω define

$$M(\omega) = \omega(0) + \omega(1) + \cdots + \omega(pf-1) \pmod{p},$$

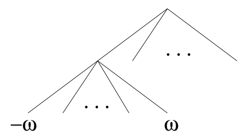
and note that $M(\omega+\omega') = M(\omega) + M(\omega') \pmod{p}$ and $M(\omega) = M(\omega^\kappa)$ (for κ see (4.4). This implies that the map $A : \mathcal{H}_{f,p} \longrightarrow \mathbb{Z}/p\mathbb{Z}$ which maps the $E_{f,p}$-symbol

$$\begin{pmatrix} u_1 & \cdots & u_s \\ b_{\omega_1} & \cdots & b_{\omega_s} \\ v_1 & \cdots & v_s \end{pmatrix}$$

to $M(\omega_1 + \cdots + \omega_s)$ is a homomorphism, which is onto, as $A(\sigma_{b_{[pf]}}) = 1$. If p is even this proves the statement, since $\sigma_{b_{[pf]}}$ is the only generator not lying in $\mathcal{H}_{f,p}'$. If, on the other hand, p is odd, then it follows from the fact that γ is an even element of $G_{n,r}$, that we can define even elements precisely as in Section 2.1. In detail: write an element h of $\mathcal{H}_{f,p}$ as a product of an order preserving element and an element with a flat $E_{f,p}$-symbol Δ, i.e., $\mathrm{top}(\Delta) = \mathrm{bot}(\Delta)$ (see Section 5.2). Hence, there is also a homomorphism from $\mathcal{H}_{f,p}$ onto the the cyclic group of order two, mapping an element onto the generator if the induced permutation in the top row of the flat symbol is odd. Since p was odd, we have now a homomorphism onto the cyclic group of order $2p$. The last assertion follows from this, and the proof is complete.

In the remainder of this section we prove that $\mathcal{H}_{f,p}$ and $\mathcal{H}_{f,p}'$ are generated by two elements. This should be compared with Section 2.6. To simplify notation, we write ω for b_ω and switch to additive notation for the group $E_{f,p}$, i.e., $b_\omega b_{\omega'}$ becomes $\omega + \omega'$ and b_ω^{-1} will be $-\omega$. We need the following a lemma.

***Lemma* 4.13** *Let z be defined by the tree diagram*



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where $\omega = [pf]$. Then ${\mathcal{H}_{f,p}}'$ is generated by ${G_{p,1}}'$ and z.

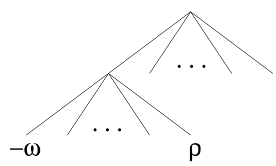
Proof. It is clear that the following relation of $E_{f,p}$-symbols holds.

$$\begin{pmatrix} u_1 & \cdots & u_s \\ \omega_1 & \cdots & \omega_s \\ v_1 & \cdots & v_s \end{pmatrix} = \begin{pmatrix} u_1 & \cdots & u_s \\ \omega_1 & \cdots & \omega_s \\ u_1 & \cdots & u_s \end{pmatrix} \begin{pmatrix} u_1 & \cdots & u_s \\ 1 & \cdots & 1 \\ v_1 & \cdots & v_s \end{pmatrix} \tag{4.8}$$

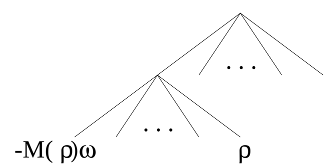
Assume that the symbol on the left hand side of (4.8) defines an element of ${\mathcal{H}_{f,p}}'$. Then $M(\omega_1 + \cdots + \omega_s) = 0$ and the element defined by the 1-symbol on the right hand side of (4.8) lies in ${G_{p,1}}'$ (see proof of Corollary 4.12). So it suffices to show that we can generate every element with an $E_{f,p}$-symbol of the form

$$\Delta = \begin{pmatrix} u_1 & \cdots & u_s \\ \omega_1 & \cdots & \omega_s \\ u_1 & \cdots & u_s \end{pmatrix},$$

with $M(\omega_1 + \cdots + \omega_s) = 0$. As in the proof of Theorem 4.10, by conjugating z with positive powers of the element α with symbol (4.7), we can generate all elements with tree diagrams of the form



where $\rho = \omega^{\kappa^i}$ for some $i \geq 0$. Observe that $\alpha \in {G_{p,1}}'$. We can now multiply some of these conjugates of z together to get every element with a tree diagram of the form



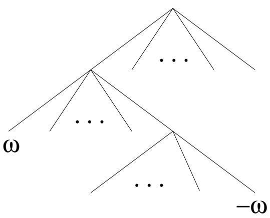
for arbitrary fp-periodic ρ. Furthermore, we can assume, by possibly conjugating, that $u_s = a_1 a_1$ in the symbol Δ. It follows that we can generate the element with symbol

$$\Gamma = \begin{pmatrix} u_1 & \cdots & u_{s-1} & a_1 a_1 \\ \omega_1 & \cdots & \omega_{s-1} & -M(\omega_1 + \cdots + \omega_{s-1})\omega \\ u_1 & \cdots & u_{s-1} & a_1 a_1 \end{pmatrix}$$

with our given generators. After multiplying Δ with Γ^{-1}, we are left to show that we can generate the element with symbol

$$\Sigma = \begin{pmatrix} u_1 & \cdots & u_{s-1} & a_1 a_1 \\ 1 & \cdots & 1 & \omega' \\ u_1 & \cdots & u_{s-1} & a_1 a_1 \end{pmatrix},$$

where $\omega' = \omega_s + M(\omega_1 + \cdots + \omega_{s-1})\omega$. Note first that $M(\omega') = 0$, and then that z is conjugate under an element of ${G_{p,1}}'$ to the element with tree diagram

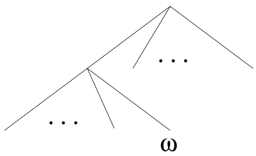


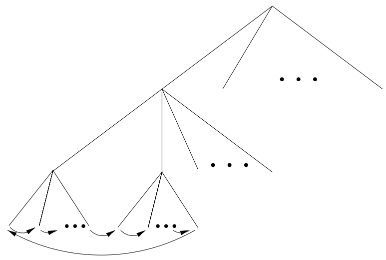
which multiplied with z gives an element conjugate to $\sigma_{\tilde{\omega}}$, where $\tilde{\omega}$ is fp-periodic with period $1 - 100 \cdots 0$. Finally, for every fp-periodic sequence $\hat{\omega}$ with $M(\hat{\omega}) = 0$, $\sigma_{\hat{\omega}}$ is a product of conjugates of $\sigma_{\tilde{\omega}}$ under positive powers of α. In particular, we get $\sigma_{\omega'}$ which is a conjugate of the element with symbol Σ, and the lemma is proved.

***Theorem* 4.14** *The groups $\mathcal{H}_{f,p}$ and ${\mathcal{H}_{f,p}}'$ are generated by two elements for all $f \geq 1$, $p \geq 2$.*

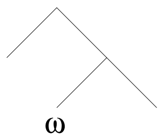
Proof. First you should fold out Appendix B again. Read those tree diagrams with the dashed lines and assume $r = p = n$. They define elements of $G_{p,1}$. Note that the results of Section 2.6 are still valid, since $G_{p,p}$ is isomorphic to $G_{p,1}$ by interpreting x_i as a_i for $1 \leq i \leq p$. For the rest of this proof $\omega = [fp]$.

We begin with $\mathcal{H}_{f,p}$. Assume $p \geq 3$, and put

$e =$ 

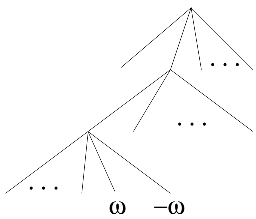
and $\tilde{a} =$  .

Check that $[\tilde{a}, e] = [\tilde{a}, b] = [b, e] = 1$. Let $x = \tilde{a}be$, then $x^p = \hat{a}b^p$ and $\langle x^{2p} \rangle = \langle b \rangle$. So, by Section 2.6, $G_{p,1} = \langle \hat{a}, b, c \rangle \subset \langle x, c \rangle$. Since $\hat{a}, b \in G_{p,1}$, $e \in \langle x, c \rangle$. But e is conjugate to σ_ω, and we are done, by Corollary 4.11. Suppose now $p = 2$, and define

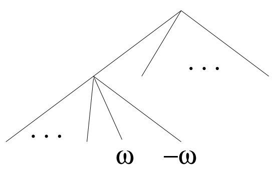
$e =$  .

Now $[e, d] = 1$ and with $x = de$, one checks that $cx^4c^{-1} = b$ and $x^6 = \hat{a}(= a)$. Hence $G_{2,1} \subset \langle x, c \rangle$. As $d \in G_{2,1}$, $e \in \langle x, c \rangle$, and e is again conjugate to σ_ω, so $\langle x, c \rangle = \mathcal{H}_{f,2}$, by Corollary 4.11.

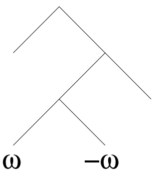
Let us now turn to $\mathcal{H}_{f,p}{}'$. Assume first that p is odd; thus, we are in the first case of the definition of b. This time define

$e =$  ,

and put $x = \bar{a}be$. Then $x^p = ab^p$ and $\langle x^{p^2} \rangle = \langle b \rangle$, whence, by Section 2.6, $G_{p,1}{}' = \langle a, b, c \rangle \subset \langle x, c \rangle$. As before, $\bar{a}, b \in G_{p,1}{}'$ implies $e \in \langle x, c \rangle$, and Lemma 4.13 completes this case, using that e is conjugate to z defined in that very lemma. Next we assume that p is even but not two. Then define

$e =$ 

and let a' be a p-th root of $\hat{a}$ such that a' fixes $a_2, \ldots, a_p$, a_1a_{p-1}, and a_1a_p, cf. Lemma 2.9. Similar as before, with $x = a'be$, we find $\langle x, c \rangle = \mathcal{H}_{f,p}{}'$. And finally, if $p = 2$, we let

$e =$ 

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and check that $a, b, c, e \in \langle de, c\rangle$, so that $\mathcal{H}_{f,2}{}'$ is generated by de and c, by Lemma 4.13. The proof of the theorem is now complete.

4.6 More Examples

Here we give some more examples of fake expansible groups. Interested readers would probably not have any difficulties in finding these very quickly on their own, as they are more or less straightforward generalisations of the groups $\mathcal{H}_{f,p}$.

First let us note that the group G_{GS} described at the end of Section 4.4 is not contained in $\mathcal{H}_{f,p}$ in any obvious way. But it is easily seen to be a special group. Let us show that a special group defined by an almost periodic sequence is also embeddable in a finitely presented simple group using groups similar to $E_{f,p}$ as follows.

Let $p \in \mathbb{N}$ and let S_p denote the symmetric group on $\{a_1, \ldots, a_p\}$. Given an infinite sequence ω with values in S_p and $i \in \{1, \ldots, p\}$, define the inescapable isomorphism $b_{i,\omega}$ by its action on the basis

$$\{{a_p}^r a_i a_1, \ldots, {a_p}^r a_i a_p, {a_p}^r a_j \,|\, r \geq 0,\ 1 \leq j \leq p,\ j \neq i\},$$

the action is

$$\begin{array}{lcll} {a_p}^r a_i a_l & \mapsto & {a_p}^r a_i (a_{l\omega(r)}), & 1 \leq i \leq p \\ {a_p}^r a_j & \mapsto & {a_p}^r a_j, & 1 \leq j \leq p-1, j \neq i. \end{array}$$

Choose a period f and define

$$E = \{b_{i,\omega_i} \,|\, 1 \leq i \leq p-1,\ \omega_i \text{ an } f\text{-periodic sequence}\}.$$

Noting that E is isomorphic to a subgroup of the direct product of $f(p-1)$ copies of S_p and hence finite, arguments similar to those in Sections 4.4 and 4.5 give the following theorem.

***Theorem* 4.15** *Let $G = \langle R, D\rangle$ be a special group as defined in Section 4.3. If all defining sequences for elements of D are periodic, then G can be embedded in a finitely presented simple group.*

The following result is slightly more interesting.

***Theorem* 4.16** *For* $1 \leq i \leq r$ *let* G_{ω_i} *be a Grigorchuk group defined by an* f_i*-periodic sequence* ω_i *with values in* $\{0, 1, \ldots, p_i\}$*. The direct product* $G_{\omega_1} \times \cdots \times G_{\omega_r}$ *is embeddable in a finitely presented simple group.*

Proof. Let $n = \sum_{i=1}^{r} p_i$ and let

$$\mathcal{A}_n = \{a_{1,1}, a_{1,2}, \cdots, a_{1,p_1}, \cdots, a_{r,1}, a_{r,2}, \cdots, a_{r,p_r}\}.$$

For $1 \leq i \leq r$ let γ_i to be the permutation of $\mathcal{A}_n$ with cycle $(a_{i,1}\, a_{i,2} \ldots a_{i,p_i})$ and let $\sigma_i = [f_i]$ (see Section 4.3 for the definition of $[f_i]$). Define e_{σ_i} to be the element whose only non-trivial columns are

$$\begin{pmatrix} a_{i,1} \\ \gamma_i \\ a_{i,1} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a_{i,p_i} \\ e_{\sigma_i^\kappa} \\ a_{i,p_i} \end{pmatrix},$$

where κ is once more the forget map (4.4). It follows as before that $\mathcal{H} = \langle G_{n,1}, e_{\sigma_i} \,|\, 1 \leq i \leq r\rangle$ is finitely presented, and that $\mathcal{H}'$ is a finitely presented simple group of finite index in $\mathcal{H}$. It is clear from the definitions that the group generated by the γ_i together with all $e_{\sigma_i}^{\alpha_i^j}$ for $0 \leq j \leq f_i - 1$ contains an isomorphic copy of $G_{\omega_1} \times \cdots \times G_{\omega_r}$, where the α_i are elements chosen in a similar way to α before and satisfy $e_{\sigma_i}^{\alpha_i} = e_{\sigma_i^\kappa}$. Observe that we have 'enough room' if $r \geq 2$ to do this for e_{σ_i} as opposed to $\sigma_{e_{\sigma_i}}$. The theorem is proved.

Chapter 5

The Order Problem

The goal of this chapter is to show that the groups $\mathcal{H}_{f,p}$ have a solvable order problem. Let us recall that a group has *solvable order problem* if given any element one can effectively decide whether it has finite or infinite order. Note that together with the solvability of the word problem this implies that we can actually find the exact order of a given element, for once we know that an element has finite order, we simply check for all its powers if they are trivial. Clearly, having a solvable order problem is passed on to subgroups, and hence ${\mathcal{H}_{f,p}}'$ has a solvable order problem.

In the first section we establish some preliminary results. Then we introduce flat symbols and step columns in Section 5.2, and combine the results to solve the order problem in Section 5.3.

5.1 The Group M

Recall from Section 4.4 the definition of the groups $E_{f,p}$ and $\mathcal{H}_{p,f}$ for $f, p \in \mathbb{N}$, and that γ is the element of $G_{p,1}$ defined by the symbol

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{p-1} & a_p \\ 1 & 1 & \cdots & 1 & 1 \\ a_2 & a_3 & \cdots & a_p & a_1 \end{pmatrix}. \tag{5.1}$$

Put $C = \langle \gamma \rangle$ and note that it is a cyclic group of order p acting naturally on $\{a_1, \ldots, a_p\}$ and hence on $\{1, \ldots, p\}$. Also, recall that every element e of $E_{f,p}$ has a symbol of the form

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{p-1} & a_p \\ \delta & 1 & \ldots & 1 & e' \\ a_1 & a_2 & \cdots & a_{p-1} & a_p \end{pmatrix}, \tag{5.2}$$

with $\delta \in C$ and $e' \in E_{f,p}$. Let M be the subgroup of $\mathcal{G}_{p,1}$ generated by $C \cup E_{f,p}$, then it follows directly from (5.1) and (5.2) that every element m of M has an M-symbol of the form

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ m_1 & m_2 & \cdots & m_p \\ a_{1\pi} & a_{2\pi} & \cdots & a_{p\pi} \end{pmatrix} \tag{5.3}$$

for some $\pi \in C$. So we can define the following homomorphism

$$\begin{array}{rccl} \Phi : & M & \longrightarrow & M \wr C \\ & m & \longmapsto & (m_1, \ldots, m_p)\pi, \end{array} \tag{5.4}$$

where m has the symbol (5.3). Since an element m of M lies in the kernel of Φ if and only if $m^\Phi = (1, \ldots, 1)1$, i.e., its symbol defines the trivial element of $\mathcal{G}_{p,1}$, we see that Φ is an embedding. Let ρ be the natural projection from $M \wr C$ onto C and define $\Psi = \Phi\rho$.

***Theorem* 5.1** *The group M as defined above has solvable word problem.*

Proof. Since M is a subgroup of the $E_{f,p}$-expansible group $\mathcal{H}_{f,p}$, M is also $E_{f,p}$-expansible. So given $m \in M$, find an $E_{f,p}$-symbol Δ for m. This can be done effectively using Proposition 3.7. Now m is trivial if and only if Δ defines the trivial element of $\mathcal{G}_{p,1}$ which happens precisely when all columns of Δ are trivial. As this can clearly be checked in finite time, the proof is complete.

***Theorem* 5.2** *The group M as defined above has solvable order problem.*

Proof. The proof is by induction on the length $l(m)$ of the element $m \in M$ as a word in the free product of the groups $E_{f,p}$ and C. Every non-trivial element of length one has order dividing p, by Lemma 4.9. Since conjugation does not change the order of an element, we can assume m to have the following form

$$\delta_1 e_1 \delta_2 e_2 \cdots \delta_n e_n, \quad \text{where } \delta_i \in C \setminus \{1\},\ e_i \in E_{f,p} \setminus \{1\},\ 1 \leq i \leq n.$$

We now consider two cases.

Case 1: $m \in \ker(\Psi)$ or equivalently $\delta_1 \delta_2 \cdots \delta_n = 1$. Then

$$m = e_1^{\delta_1^{-1}} e_2^{(\delta_1\delta_2)^{-1}} \cdots e_{n-1}^{(\delta_1 \cdots \delta_{n-1})^{-1}} e_n$$

and m^Φ is an element of M^p, the base group of $M \wr C$. Suppose $m^\Phi = (m_1, \ldots, m_p)$. It follows from (5.2) that each factor $e_i^{(\delta_1 \cdots \delta_i)^{-1}}$ contributes at most one letter each to m_i and hence $l(m_i) < l(m)$ for $1 \leq i \leq p$. By the induction hypothesis, we can decide the order problem for each m_i, and therefore, the order problem for m. For it is clearly that m has finite order if and only if each m_i has finite order.

Case 2: $m \notin \ker(\Psi)$ or equivalently $\delta_1\delta_2\cdots\delta_n = \delta \neq 1$. Let q be the order of δ. Then $m^q \in \ker(\Psi)$ and

$$m^q = \prod_{i=0}^{q-1}(e_1^{\delta_1^{-1}} e_2^{(\delta_1\delta_2)^{-1}} \cdots e_{n-1}^{(\delta_1\cdots\delta_{n-1})^{-1}} e_n^{\delta^{-1}})^{\delta^{-i}}.$$

Suppose $m^q = (m_1, \ldots, m_p)$. If $q \neq p$, then the two non-trivial components of $e_i^{(\delta_1\cdots\delta_i)^{-1}}$ lie in different δ-orbits. This in turn implies that each factor $e_i^{(\delta_1\cdots\delta_i)^{-1}}$ together with all its conjugates under δ^j, $0 \leq j \leq q-1$, contributes at most one letter to each m_j, $1 \leq j \leq p$. Hence $l(m_j) < l(m)$ for $1 \leq j \leq p$ and we can refer to the induction hypothesis.

Assume now that $q = p$ and $m^p = (m_1, \ldots, m_p)$. From (5.2) we get that each $e_i^{(\delta_1\cdots\delta_i)^{-1}}$ together with all its conjugates under δ^j, $1 \leq j \leq p-1$, can contribute at most two letters to m_k for $1 \leq k \leq p$. Hence $l(m_k) \leq 2n = l(m)$. Note that all the m_k are pairwise conjugate so that it suffices to decide the order problem for any of the m_k in order to solve the order problem for m. If for some k, $1 \leq k \leq p$, $l(m_k) < 2n$ we plainly refer to the induction hypothesis.

So let us speak of the *worst case* if actually $l(m_k) = l(m)$ for $1 \leq k \leq p$. In this case we apply the whole procedure to m_1 instead of m. If we find ourselves again in the worst case, we obtain an m_2, a component of m_1^p of the same length and so forth.

Suppose we always stay in the worst case and consider the resulting sequence of elements m_i, $i \geq 0$, arising as described above. Observe that each m_i is conjugate (via an element of $E_{f,p} \cup C$) to an element of the form

$$\epsilon_1 f_1 \cdots \epsilon_n f_n, \tag{5.5}$$

where $\epsilon_i \in C \setminus \{1\}$ and $f_i \in E_{f,p} \setminus \{1\}$. But there are only finitely many elements of the form (5.5). This means $m_s^x = m_r$ for some $s < r \in \mathbb{N}$ and some $x \in M$, and we will find this after at most $K = (p-1)^n(p^{fp}-1)^n$ steps. It follows from the procedure that $m_s^{p^{r-s}}$ and m_s have the same order. Hence, if m_s is not trivial, m_s and m have infinite order, and m has finite order, if m_s is trivial. As we can decide whether m_s is trivial or not, by Theorem 5.1, this solves the order problem for m, and the proof is complete.

5.2 STEP COLUMNS AND FLAT SYMBOLS

Here begins our investigation of particularly nice symbols. Flat symbols play also a major role in Chapter 6.

Let Δ be a symbol and let $\mathcal{C}$ be a column of Δ. We call $\mathcal{C}$ a *step column* if it is of the form

$$\begin{pmatrix} b \\ g \\ bv \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} bv \\ g \\ b \end{pmatrix}$$

for some $v \in W_n \setminus \{\emptyset\}$. Let M be defined as in the previous section. The next lemma shows that for an element g of $\mathcal{H}_{f,p}$ either all its M-symbols have a step column or none of its M-symbols has a step column.

***Lemma* 5.3** *Suppose the M-symbol Δ has a step column. Then every expansion of Δ has a step column, and every contraction of Δ which itself is an M-symbol has a step column.*

Proof. By possibly replacing Δ by Δ^{-1}, we may assume that

$$\mathcal{C} = \begin{pmatrix} b \\ m \\ bv \end{pmatrix}$$

is the step column of Δ.

We deal with expansions first. It certainly suffices to prove the statement for simple expansions, and we only have to consider the case that $\mathcal{C}$ gets expanded. Assuming that $(m_1, \ldots, m_p)\pi$ is the image of m under the embedding Φ (see (5.4)), $\mathcal{C}$ gets replaced by

$$\begin{pmatrix} ba_1 & \cdots & ba_p \\ m_1 & \cdots & m_p \\ bva_{1\pi} & \cdots & bva_{p\pi} \end{pmatrix}.$$

Hence, if $v = a_i w$, then

$$\begin{pmatrix} ba_i \\ m_i \\ ba_i(wa_{i\pi}) \end{pmatrix}$$

is the step column of the expansion.

To prove the statement for contractions we may restrict ourselves to simple contractions which involve the column $\mathcal{C}$. Assume

$$\mathcal{C} = \begin{pmatrix} b \\ m \\ bv \end{pmatrix} = \begin{pmatrix} b'a_i \\ m \\ bv'a_{i'} \end{pmatrix}.$$

Then Δ must also contain columns

$$\begin{pmatrix} b'a_j \\ m_j \\ bv'a_{j\pi} \end{pmatrix}$$

for $1 \leq j \leq p$, $j \neq i$, and some permutation π of $\{a_1, \ldots, a_n\}$ with $i\pi = i'$. Moreover, there must be $m' \in M$ such that its image under the embedding Φ is $(m_1, \ldots, m_{i-1}, m, m_{i+1}, \ldots, m_p)\pi$ for the resulting contraction, Δ' say, to be an M-symbol. But then

$$\begin{pmatrix} b' \\ m' \\ b'a_iv' \end{pmatrix} \in \Delta'$$

which is a step column as $a_iv' \neq \emptyset$ and the lemma is proved.

A symbol Δ is called *flat* if $\mathrm{top}(\Delta) = \mathrm{bot}(\Delta)$. Observe, that if g is the element defined by the flat symbol Δ, the combination Δ^t exists for all $t \in \mathbb{Z} \setminus \{0\}$ and $\mathrm{top}(\Delta)$ is a $\langle g\rangle$-basis in the sense of Section 4.2. This motivates the following definition(s) in which $\mathcal{T}$ is to be replaced by $E_{f,p}$, fake-$E_{f,p}$, or M. Let $g \in \mathcal{H}_{f,p}$. We call Δ a *$\langle g\rangle$-$\mathcal{T}$-symbol* if Δ is a flat $\mathcal{T}$-symbol for g. We also say *g has a flat $\mathcal{T}$-symbol* if there exists a $\langle g\rangle$-$\mathcal{T}$-symbol. Let us emphasise that being a $\langle g\rangle$-$\mathcal{T}$-symbol means in particular being a $\mathcal{T}$-symbol for g, rather than just any element of $\langle g\rangle$.

***Lemma* 5.4** *Let $g \in \mathcal{H}_{f,p}$, then g has a flat fake-$E_{f,p}$-symbol if and only if it has a flat M-symbol.*

Proof. The *only if* part is trivial. Let us proceed with the *if* part. For $k \in \mathbb{N}$ let L_k denote the set of all words of length k in W_n. The lemma is now an immediate consequence of

***Lemma* 5.5** *For every $m \in M$ there exists $K \in \mathbb{N}$ so that for all $k \geq K$, m has a flat fake-$E_{f,p}$-symbol Δ with* $\mathrm{top}(\Delta) = L_k$.

Proof. It suffices to prove that there exists $K \in \mathbb{N}$ such that m has a flat fake-$E_{f,p}$-symbol Δ with $\mathrm{top}(\Delta) = L_K$. For then, the result of simply expanding every column of Δ is a $\langle m\rangle$-fake-$E_{f,p}$-symbol whose top row is equal to L_{K+1}, by (5.1) and (5.2).

The proof is now by induction on the length $l(m)$ of m with respect to the generating set $C \cup E_{f,p}$. By (5.1) and (5.2) again, the statement surely holds if $l(m) \leq 2$ with $K = 1$. So assume $l(m) > 2$ and write $m = m'x$, where $x \in C \cup E_{f,p}$ and $l(m') < l(m)$. Let Γ be the $\langle m'\rangle$-fake-$E_{f,p}$-symbol with

$\mathrm{top}(\Gamma) = L_K$ for suitable K whose existence is guaranteed by the induction hypothesis, and let Σ be a $\langle x \rangle$-fake-$E_{f,p}$-symbol with $\mathrm{top}(\Sigma) = L_K$. Then $\Gamma\Sigma$ exists and its middle row entries are elements of M of length at most two. According to what was said above, the result of simply expanding each column of $\Gamma\Sigma$ is a $\langle m \rangle$-fake-$E_{f,p}$-symbol Δ with $\mathrm{top}(\Delta) = L_{K+1}$. This establishes Lemmas 5.5 and 5.4.

Let us also record the following useful criterion for finding a flat symbol. Recall that for a finite family $(B_i)_{i \in I}$ of bases of W_n, $\bigsqcup_{i \in I} B_i$ denotes the basis of the inescapable subspace $\bigcap_{i \in I} B_i W_n$ of W_n.

***Lemma* 5.6** *In the following $\mathcal{T}$ may be replaced by $E_{f,p}$, fake-$E_{f,p}$ or M. Let $g \in \mathcal{H}_{f,p}$, $0t \in \mathbb{N}$, and assume that $\Delta_1, \ldots, \Delta_t$ are $\mathcal{T}$-symbols for g such that the combination $\Delta_1 \cdots \Delta_t$ exists. If Γ is the symbol for g with $\mathrm{top}(\Gamma) = \bigsqcup_{i=1}^{t} \mathrm{top}(\Delta_i)$, then Γ is a $\mathcal{T}$-symbol and Γ is flat if and only if $\mathrm{top}(\Delta_1) = \mathrm{bot}(\Delta_t)$.*

Proof. Put $B = \bigsqcup_{i=1}^{t} \mathrm{top}(\Delta_i)$, let Γ be the symbol for g with $\mathrm{top}(\Gamma) = B$ and let

$$\begin{pmatrix} b \\ x \\ c \end{pmatrix}$$

be a column of Γ. By Lemma 1.1 b) and Lemma 1.4, this is also a column of one of the given symbols, say of Δ_j, so in particular Γ is a $\mathcal{T}$-symbol and $c \in \mathrm{bot}(\Delta_j)$. Since $b \in B$, we have $b \in \mathrm{top}(\Delta_i) W_p$ for each i, so that g has the columns

$$\begin{pmatrix} b_i v_i = b \\ x_i \\ w_i \end{pmatrix},$$

where $b_i \in \mathrm{top}(\Delta_i)$, and consequently $w_i \in \mathrm{bot}(\Delta_i) W_p$. By Lemma 1.4 again, all w_i are equal to c, whence $c \in \bigcap_{i=1}^{t} \mathrm{bot}(\Delta_i) W_p$. It follows from Lemma 1.1 c) that $c \in \bigsqcup_{i=1}^{t} \mathrm{bot}(\Delta_i)$. Assuming $\mathrm{top}(\Delta_1) = \mathrm{bot}(\Delta_t)$, it follows that $B = \bigsqcup_{i=1}^{t} \mathrm{bot}(\Delta_i)$, and hence $c \in B$, so Γ is flat. On the other hand, if Γ is flat, we have $B = \mathrm{bot}(\Gamma) \subset \bigsqcup_{i=1}^{t} \mathrm{bot}(\Delta_i)$, which implies $B = \bigsqcup_{i=1}^{t} \mathrm{bot}(\Delta_i)$, as both, B and $\bigsqcup_{i=1}^{t} \mathrm{bot}(\Delta_i)$, are bases. This proves the lemma.

An easy consequence of this is a characterisation of elements of finite order in $\mathcal{H}_{f,p}$.

***Proposition* 5.7** *The element g of $\mathcal{H}_{f,p}$ has finite order if and only if it has a flat $E_{f,p}$-symbol.*

Proof. If Δ is a flat $E_{f,p}$-symbol for g then it is clear that $g^{s!p} = 1$, where s is the number of columns of Δ. This is because $E_{f,p}$ is an abelian group of exponent p (Lemma 4.9). For the converse assume that $g^t = 1$ and we are given $E_{f,p}$-symbols $\Delta_1, \dots, \Delta_t$ for g such that the combination $\Delta_1 \cdots \Delta_t$ exists ($\mathcal{H}_{f,p}$ is $E_{f,p}$-expansible, by Lemma 4.9). This combination is a symbol for the trivial element and thus $\mathrm{top}(\Delta_1) = \mathrm{bot}(\Delta_t)$. By Lemma 5.6, the symbol Γ for g with $\mathrm{top}(\Gamma) = \bigsqcup_{i=1}^{t} \mathrm{bot}(\Delta_i)$ is a $\langle g \rangle$-$E_{f,p}$-symbol, and the proposition is proved.

5.3 Solving the Order Problem

Our aim in this section is to establish the following dichotomy.

***Proposition* 5.8** *For $g \in \mathcal{H}_{f,p}$ one of the following holds.*

(i) There is a flat fake-$E_{f,p}$-symbol for g, or

(ii) for some $t \in \mathbb{N}$, g^t has a step column.

First we show that it implies

***Theorem* 5.9** *The order problem for $\mathcal{H}_{f,p}$ is solvable.*

Proof. First we show that we can decide in finite time which of the two cases of the proposition holds. So let $g \in \mathcal{H}_{f,p}$ be given. If it is given to us as a word in the generators, we can effectively compute an M-symbol for g, since we can execute simple expansions effectively by (4.6). So we might as well assume that we are given an M-symbol Δ. In case g satisfies (i), g also has a flat M-symbol Γ such that Γ is an expansion of Δ, by Lemmas 5.4 and 6.1. Hence, checking all expansions of Δ, say simple expansions first, then simple expansions of the simple expansions an so forth, will eventually reveal a flat M-symbol. If, on the other hand, g satisfies (ii), then the following test works: compute symbols $\Delta_2, \Delta_3, \dots$ for $g^2, g^3, \dots$ and check them for step columns. By Lemma 5.3, it does not matter which particular M-symbol Δ_i for g^i we have computed, so the procedure stops. Thus, running both tests simultaneously, one of them will stop, by the proposition, as required.

To complete the proof we note first that an element g with a step column has infinite order, for otherwise there would be a flat $E_{f,p}$-symbol for g, by Lemma 5.7, which contradicts Lemma 5.3, by which every symbol for g has a step column. Now suppose g has a flat fake-$E_{f,p}$-symbol Δ. Then for every column of Δ^t the top and bottom row entries are equal, where $t = |\mathrm{top}(\Delta)|!$. So g has finite order if and only if all the middle row entries of Δ^t are periodic which we can decide by Theorem 5.2. This completes the proof of the Theorem.

The proof of the proposition that follows is close to Higman's proof for the solvability of the order problem for $G_{n,r}$. But to make those arguments work we need to extend the domain of the elements of $E_{f,p}$. As mentioned before, the group $E_{f,p}$ can be described as a group of tree automorphisms of the tree W_p, and we adopt this point of view here. More precisely, for $e \in E_{f,p}$ and $w \in W_p$ define

$$w^{\bar{e}} = \begin{cases} w^e, & \text{if } w^e \text{ is defined} \\ w, & \text{otherwise} \end{cases}$$

A glance at the definition of $e(= b_\omega)$ in Section 4.3 shows that $e \longmapsto \bar{e}$ is a well define injection from the group $E_{f,p}$ of maximal inescapable isomorphisms into the automorphism group of the tree W_p (see Section 1.5). Moreover, this definition is compatible with expansions in the sense that for every fake-$E_{f,p}$-symbol Δ for e the restriction of $\bar{e}$ to the subspace $\text{top}(\Delta)W_p$ is the same map as the one defined by the symbol obtained from Δ by 'barring' all middle row entries. Here we are assuming that we have also extended the action of γ by setting $\emptyset^{\bar{\gamma}} = \emptyset$.

For the remainder of this section we omit the bars and assume that elements of M act as tree automorphisms.

Remark. The point is that, contrary to the situation for inescapable isomorphisms, $(bw)^g$ is now defined for all $w \in W_p$ and $b \in \text{top}(\Delta)$, where Δ is any M-symbol for g. But it is no longer true that for $v = wz$, $(bv)^g = (bw)^g z$. However, we still have $(bv)^g = (bw)^g z'$ for some $z' \in W_p$ of the same length as z, because middle row entries preserve this length (automorphisms of a rooted tree have to fix the root).

From now on g is an arbitrary but fixed element of $\mathcal{H}_{f,p}$. We call a basis B *benign (for g)* if it is finite and $w^{g^{-1}}$ and w^g are defined for all $w \in BW_p$. Note that every expansion of a benign is also benign.

***Lemma* 5.10** *There exists a benign basis B such that, for every $w \in BW_p$ there is $\varepsilon \in \{\pm 1\}$ with $w^{g^{\varepsilon t}} \in BW_p$ for all $t \in \mathbb{N}$.*

Proof. Let B be a benign basis and put $U = BW_p$. For $u \in U$ define the *quasi-orbit* $O(u)$ of u as follows. Let $s \leq 0$ be minimal subject to $u^{g^k} \in U$ for all k with $s \leq k \leq 0$ or $s = -\infty$ if $u^{g^k} \in U$ for all $k \leq 0$, and let $t \geq 0$ be maximal subject to $u^{g^k} \in U$ for all k with $0 \leq k \leq t$ or $t = \infty$ if $u^{g^k} \in U$ for all $k \geq 0$. Then $O(u) = \{u^{g^k} \mid s \leq k \leq t\}$. We say u is of *finite type* if both s and t are finite, and in that case u^{g^s} and u^{g^t} are called *ends* of $O(u)$. Observe that B satisfies the lemma if there are no elements in U of finite type and that $O(v) = O(u)$ for all $v \in O(u)$.

Let us show that there are only finitely many elements of finite type in U. Let v be an end of $O(u)$, then $v^{g^\varepsilon} \in W_p \setminus U$ for some $\varepsilon \in \{\pm 1\}$ and we say

v^{g^ε} is an *exit* of $O(u)$. It is straightforward that each element of $W_p \setminus U$ is the exit of at most two different quasi-orbits. Since $W_p \setminus U$ is finite and every quasi-orbit of finite type is finite and has an exit, there are only finitely many elements of finite type in U.

Now assume that u is of finite type and write $u = bv$ with $b \in B$ and $v \in W_p$. Then b is also of finite type, since $b^{g^k} \leq u^{g^k}$ for all k such that b^{g^k} is defined. Let C be the simple expansion of B at b and note that CW_p has the same elements of finite type as U but b. Hence, by induction on the number of elements of finite type, there is a benign basis D such that D has no elements of finite type, and the lemma is proved.

A benign basis satisfying Lemma 5.10 is called a *normal basis (for g)*. Observe that the proof of the lemma gives an effective procedure for finding a normal basis and, moreover, that every benign basis has a normal expansion.

Proof of Proposition 5.8. Let B be a normal basis for g such that g has an M-symbol with top row B. Suppose $b \in B$ such that $O(b)$ finite. Then we claim that $b^{g^i} \in B$ for all $i \in \mathbb{Z}$. For otherwise there is $b' \in O(b)$ such that $b'^g = b''v$ with $b'' \in B$ and $\emptyset \neq v \in W_p$, and hence $b' = (b''v)^{g^{-1}} = b''^{g^{-1}}v'$ for some $\emptyset \neq v' \in W_p$, since B is benign. This implies $b''^{g^{-1}} \notin BW_p$ which force $O(b'')$ to be infinite, as B is normal. It follows that $O(b)$ must also be infinite, a contradiction. Thus, the M-symbol for g with top row B is flat if all elements of B have finite quasi-orbits. So (i) holds , by Lemma 5.4.

Now assume that $O(b)$ is infinite for $b \in B$. By possibly replacing g by g^{-1} we can assume that $b^{g^i} \in BW_p$ for all $i \geq 0$. Define $t(b)$ to be the maximal integer subject to $b^{g^i} \notin B$ for $0 \leq i < t(b)$ and let $\hat{b} = b^{g^{t(b)}}$ ($t(b)$ exists since B is finite). Then $\hat{b} = b_1 v_1$ for some $b_1 \in B$ and $\emptyset \neq v_1 \in W_p$. As before, $b_1^{g^{-1}} \notin BW_p$, and $O(b_1)$ is infinite, since it is not of finite type (B is normal). In particular, $b_1^{g^i} \in BW_p$ for $i \geq 0$. Iterating this argument we get $b_i \in B$ with $b_i v_i = \widehat{b_{i-1}}$, $v_i \in W_p$, and, as B is finite, we find $j \geq 0$ with $j < k$ and $b_j = b_k$, where $b_0 = b$. For $1 \leq i \leq k-1$ put $t_i = t(b_i)$ and check that

$$\begin{aligned} b_j^{g^{t_j + t_{j+1} + \cdots + t_{k-1}}} &= (b_{j+1}*)^{g^{t_{j+1}+\cdots+t_{k-1}}} \\ &= (b_{j+2}*)^{g^{t_{j+2}+\cdots+t_{k-1}}} \\ &\vdots \\ &= (b_{k-1}*)^{g^{t_{k-1}}} \\ &= b_j* \end{aligned}$$

where the $*$ is to be replaced by some non-empty element of W_p, possibly varying from line to line. It is now clear that $g^{t_j+t_{j+1}+\cdots+t_{k-1}}$ has a step column, and the proof of the proposition is complete.

As all the steps of this proof can be carried out effectively, we get immediately

***Corollary* 5.11** *For $g \in \mathcal{H}_{f,p}$ it is effectively decidable whether g has a flat fake-$E_{f,p}$-symbol.*

Chapter 6

The Conjugacy Problem

This chapter is the outcome of an attempt to solve the conjugacy problem for the groups $\mathcal{H}_{p,f}$. A group has a *solvable conjugacy problem* if given two arbitrary elements one can effectively decide whether they are conjugate in the group. Since an element is conjugate to the identity if and only if it is the identity, the solvability of the conjugacy problem implies the solvability of the word problem. However, we cannot give a complete solution but we prove an affirmative answer for elements that have flat symbols. This includes all elements of finite order (Proposition 5.7) and in addition some elements of infinite order. Since this involves already quite a lot of notation even in the case where p is a prime and gets considerably more complicated if p is not prime, we assume from now on that p is a prime. This restriction is most important in Section 6.6.

6.1 Pseudo-Orbits and their Characteristics

Here we investigate flat fake-$E_{f,p}$-symbols in greater detail. Although this notion came up earlier, it was mainly developed to treat the conjugacy problem, and hence it appears in this chapter.

Let Δ be a $\langle g\rangle$-fake-$E_{f,p}$-symbol (this includes being a $\langle g\rangle$-$E_{f,p}$-symbol). Then there is a well defined permutation $\pi_g(\Delta)$ of $\operatorname{top}(\Delta)$, given by $b^{\pi_g(\Delta)} = c$, where

$$\begin{pmatrix} b \\ x \\ c \end{pmatrix}$$

is a column of Δ. We call the orbits of $\pi_g(\Delta)$ in $\operatorname{top}(\Delta)$ *pseudo-orbits of* g. We also let $\pi_g(\Delta)$ permute (the set of columns of) Δ according to their top-entries. For convenience, we think hereafter of a symbol as a set of columns, so that we may speak of the *pseudo-orbits of g in* Δ. Since such a pseudo-orbit $\mathcal{O}$ is again nothing but a set of columns, we will usually denote it by

$$\begin{pmatrix} o_1 & o_2 & \cdots & o_{t-1} & o_t \\ x_1 & x_2 & \cdots & x_{t-1} & x_t \\ o_2 & o_3 & \cdots & o_t & o_1 \end{pmatrix}. \tag{6.1}$$

Let $PS(\Delta)$ denote the set of pseudo-orbits of Δ, having in mind that this requires Δ to be at least a flat symbol. But we will only deal with flat M-symbols in the sequel.

Convention. It is now convenient to distinguish a word over the alphabet $C \cup E_{f,p}$ from the element of M it represents. We have to be even more rigorous, and take elements of the free product $C * E_{f,p}$ into account as well. If $w \equiv x_1 \cdots x_r$ is a word over $C \cup E_{f,p}$, i.e., $x_i \in C \cup E_{f,p}$, we write $\overline{w}$ for the element of M it represents. By $l(w)$ we denote the length of w as an element of $C * E_{f,p}$, i.e., we lump together consecutive occurrences of letters of the same factor and.

Suppose now $\mathcal{O} \in PS(\Delta)$ is given by (6.1). We define the *characteristic of* $\mathcal{O}$, denoted $\chi(\mathcal{O})$, to (be the word) $x_1 x_2 \cdots x_t$. And define the *trace of the pseudo-orbit* $\mathcal{O}$by

$$\operatorname{tr}(\mathcal{O}) = \begin{cases} 0, & \text{if } \overline{x_1 x_2 \cdots x_t} = 1 \\ 1, & \text{otherwise.} \end{cases}$$

Note that the definition of $\chi(\mathcal{O})$ depends on the choice of o_1 and that, for a different choice of o_1 the characteristic is a cyclic conjugate $x_i \cdots x_t x_1 \cdots x_{i-1}$ of $\chi(\mathcal{O})$. In contrast, the characteristic is independent of that choice. It also makes sense to say that $\mathcal{O}$ has *finite (infinite)* characteristic if $\overline{\chi(\mathcal{O})}$ has finite (infinite) order. Similarly, we say that $\chi(\mathcal{O})$ *lies in the kernel of* Ψ if $\overline{\chi(\mathcal{O})}$ does, where Ψ is the homomorphism defined just before Theorem 5.1. It is clear that all these latter definitions are independent of the choice of o_1. But they depend heavily on the intrinsic order that is determined by the cycle structure of $\pi_g(\Delta)$.

Let $\mathcal{O}$ be a pseudo-orbit of Δ. By a *simple pseudo-orbit expansion of* Δ *at* $\mathcal{O}$ we mean the resulting symbol after simply expanding Δ at all columns of $\mathcal{O}$ (a subset of Δ). And as before, we say that Γ is a *pseudo-orbit expansion* of Δ if there are finitely many symbols $\Delta_0, \ldots, \Delta_k$ such that $\Delta = \Delta_0$, $\Gamma = \Delta_k$ and Δ_i is a simple pseudo-orbit expansion of Δ_{i-1} at some $\mathcal{O} \in PS(\Delta_{i-1})$ for $1 \leq i \leq k$.

***Lemma* 6.1** *Suppose $g \in \mathcal{H}_{f,p}$ has a flat fake-$E_{f,p}$-symbol. Then g has a unique minimal flat fake-$E_{f,p}$-symbol Δ, in the sense that every other $\langle g \rangle$-$E_{f,p}$-symbol Γ is an expansion of Δ. Moreover, every such Γ is in fact a pseudo-orbit expansion of Δ, and conversely, every pseudo-orbit expansion of Δ gives such a Γ. Consequently, g has arbitrary large flat fake-$E_{f,p}$-symbols.*

Proof. Suppose Γ_1 and Γ_2 are two $\langle g\rangle$-fake-$E_{f,p}$-symbols and $\mathrm{top}(\Gamma_1) \neq \mathrm{top}(\Gamma_2)$. We may now assume that there is $o_1 \in \mathrm{top}(\Gamma_1)$ with $o_1 \notin \mathrm{top}(\Gamma_2)W_p$. Let $\mathcal{O}$ be the pseudo-orbit in Γ_1 containing o_1. Then it is clear that for every $o \in \mathrm{top}(\mathcal{O})$, $o \notin \mathrm{top}(\Gamma_2)W_p$. Let

$$\mathcal{U} = \{\mathcal{C} \in \Gamma_2 \mid \mathrm{top}(\mathcal{C}) \in \mathrm{top}(\Gamma_2) \cap \mathrm{top}(\mathcal{O})W_p\}$$

and observe that $\Gamma_3 = (\Gamma_2 \backslash \mathcal{U}) \cup \mathcal{O}$ is a $\langle g\rangle$-fake-$E_{f,p}$-symbol and a proper contraction of Γ_2. Therefore, the assumption that both Γ_1 and Γ_2 are minimal gives a contradiction, and hence there is a unique minimal flat fake-$E_{f,p}$-symbol for g. The second part of the lemma follows from the obvious fact that for every proper non-empty subset J of $\{1, \ldots, t\}$ the result of simply expanding all columns with top entry o_j, $j \in J$, of the pseudo-orbit (6.1) does not lead to a flat fake-$E_{f,p}$-symbol. This proves the lemma.

6.2 Conjugate Flat Symbols

Here we gather some obviously necessary conditions for elements with flat fake-$E_{f,p}$-symbols to be conjugate. Assume that g and h are elements of a single conjugacy class of $\mathcal{H}_{f,p}$, say $x^{-1}gx = h$ for some $x \in \mathcal{H}_{f,p}$, and that g has a flat fake-$E_{f,p}$-symbol. Since $\mathcal{H}_{f,p}$ is $E_{f,p}$-expansible (Lemma 4.9) we find $E_{f,p}$-symbols Σ, Δ, Λ, and Γ for x^{-1}, g, x, and h, respectively, such that

$$\Sigma\Delta\Lambda = \Gamma. \tag{6.2}$$

Recall that each (simple) expansion of a fake-$E_{f,p}$-symbol is again a fake-$E_{f,p}$-symbol and, that for every combination $\Delta_1 \cdots \Delta_r$ of fake-$E_{f,p}$-symbols and given expansion Γ_i of Δ_i, there are expansions $\Gamma_j \succeq \Delta_j$, $1 \leq j \neq i \leq r$, so that $\Gamma_1 \cdots \Delta_r$ exists, by (5.1) and (5.3). By our assumptions, it is now possible to do the following: expand (6.2) so that the symbol for g becomes a flat fake-$E_{f,p}$-symbol. The obtained combination is of the form

$$\Sigma^{-1}\Delta\Sigma = \Gamma, \tag{6.3}$$

where Γ is a fake-$E_{f,p}$-symbol for h, Δ is a $\langle g\rangle$-fake-$E_{f,p}$-symbol, and Σ is a fake-$E_{f,p}$-symbol for x. It follows that Γ is in fact a flat fake-$E_{f,p}$-symbol. Let (6.1) be a pseudo-orbit, $\mathcal{O}$ say, of g in Δ and let

$$\begin{pmatrix} w_1 & w_2 & \cdots & w_{t-1} & w_t \\ y_1^{-1} & y_2^{-1} & \cdots & y_{t-1}^{-1} & y_t^{-1} \\ o_1 & o_2 & \cdots & o_{t-1} & o_t \\ x_1 & x_2 & \cdots & x_{t-1} & x_t \\ o_2 & o_3 & \cdots & o_t & o_1 \\ y_2 & y_3 & \cdots & y_t & y_1 \\ w_2 & w_3 & \cdots & w_t & w_1 \end{pmatrix}$$

be the corresponding part of the left hand side of (6.3). Then

$$\begin{pmatrix} w_1 & w_2 & \cdots & w_{t-1} & w_t \\ y_1^{-1}x_1y_2 & y_2^{-1}x_2y_3 & \cdots & y_{t-1}^{-1}x_{t-1}y_t & y_t^{-1}x_ty_1 \\ w_2 & w_3 & \cdots & w_t & w_1 \end{pmatrix} \tag{6.4}$$

is a pseudo-orbit, $\mathcal{U}$ say, of h in Γ. Moreover, the characteristic of $\mathcal{U}$ is $y_1^{-1}\chi(\mathcal{O})y_1$. In particular, $\mathcal{U}$ has finite characteristic or lies in the kernel of Ψ if and only if $\mathcal{O}$ has finite characteristic or respectively lies in the kernel of Ψ. Also note that the y_i are elements of the finite set $E_{f,p} \cup C$. Thus, given a flat fake-$E_{f,p}$-symbol Ω for g, we can list in finite time all possible characteristics of pseudo-orbits of elements conjugate to g such that Ω occurs in the corresponding combination of fake-$E_{f,p}$-symbols. This should make clear that we have to investigate the behaviour of pseudo-orbits under pseudo-orbit expansions if we seek a solution to the conjugacy problem. This is precisely what follows.

6.3 Pseudo-Orbit Expansions

Let Δ be a flat fake-$E_{f,p}$-symbol and let $\mathcal{O} \in PS(\Delta)$. We say that $\mathcal{O}$ is *of class I* if $\chi(\mathcal{O})$ lies in the kernel of the homomorphism $\Psi : M \longrightarrow C$ defined just before Theorem 5.2, and of *class II* otherwise.

***Lemma* 6.2** *Let Δ be a $\langle g\rangle$-fake-$E_{f,p}$-symbol and $\mathcal{O} \in PS(\Delta)$. Suppose Γ is the simple pseudo-orbit expansion of Δ at $\mathcal{O}$ and $\chi(\mathcal{O})^\Phi = (m_1, \ldots, m_p)\pi$. Then we can order the columns of Γ so that Γ has the same pseudo-orbits as Δ except for $\mathcal{O}$ which gets replaced by pseudo-orbits $\mathcal{O}_i$ according to the following two cases.*

1. *If $\mathcal{O}$ is of class I, then $1 \leq i \leq p$, $\chi(\mathcal{O}_i) = m_i$, and $|\mathcal{O}_i| = |\mathcal{O}|$.*
2. *If $\mathcal{O}$ is of class II, then $i = 1$, $\chi(\mathcal{O}_1) = m_1 m_{1\pi} \cdots m_{1\pi^{p-1}}$, and $|\mathcal{O}_1| = p|\mathcal{O}|$.*

Moreover, $l(\chi(\mathcal{O}_i)) \leq l(\chi(\mathcal{O}))$ for all i in question and with strict inequality in the first case.

Proof. Suppose $\mathcal{O}$ is given by (6.1) and $x_i^\Phi = (x_{i,1}, \ldots, x_{i,p})\sigma_i$ for $1 \leq i \leq p$. Put $\pi_i = \sigma_1 \cdots \sigma_i$, $1 \leq i \leq t$ and observe that $\pi = \pi_t$, so $\pi_t = 1$ if and only if $\mathcal{O}$ is of class I.

Suppose first that $\mathcal{O}$ is of class I. Then it follows from the correspondence between Φ and symbols for elements of M (see (5.3) and (5.4)) that the result of simply expanding all columns of $\mathcal{O}$ contains

$$\begin{pmatrix} o_1a_i & o_2a_{i\pi_1} & \cdots & o_ta_{i\pi_{t-1}} \\ x_{1,i} & x_{2,i\pi_1} & \cdots & x_{t,i\pi_{t-1}} \\ o_2a_{i\pi_1} & o_3a_{i\pi_2} & \cdots & o_1a_i \end{pmatrix}$$

for $1 \leq i \leq p$. This is clearly a pseudo-orbit, $\mathcal{O}_i$ say, and its characteristic is $x_{1,i}x_{2,i\pi_1}\cdots x_{t,i\pi_{t-1}}$ which is soon shown to be the i-th component of $(x_1x_2\cdots x_t)^\Phi$. Clearly, $|\mathcal{O}_i| = |\mathcal{O}|$.

Assume now that $\mathcal{O}$ is of class II, i.e., $\pi_t \neq 1$. Then the result of simply expanding all columns of $\mathcal{O}$ is

$$\begin{pmatrix} o_1a_1 & o_2a_{1\pi_1} & \cdots & o_ta_{1\pi_{t-1}} & \cdots & o_1a_{i\pi_t^{p-1}} & \cdots & o_ta_{1\pi_t^{p-1}\pi_{t-1}} \\ x_{1,1} & x_{2,1\pi_1} & \cdots & x_{t,1\pi_{t-1}} & \cdots & x_{1,1\pi_t^{p-1}} & \cdots & x_{t,1\pi_t^{p-1}\pi_{t-1}} \\ o_2a_{1\pi_1} & o_3a_{1\pi_2} & \cdots & o_1a_{1\pi_t} & \cdots & o_2a_{1\pi_t^{p-1}\pi_1} & \cdots & o_1a_1 \end{pmatrix},$$

which is a pseudo-orbit of cardinality $p|\mathcal{O}|$ and with characteristic

$$x_{1,1}x_{2,1\pi_1}\cdots x_{t,1\pi_{t-1}}\cdots x_{1,1\pi_t^{p-1}}\cdots x_{t,1\pi_t^{p-1}\pi_{t-1}}.$$

The claims about the relations between the characteristics of $\mathcal{O}$ and its expansion follow immediately from our assumption on x_i^Φ and the following computation.

$$\begin{aligned}(x_1x_2\cdots x_t)^\Phi &= (x_{1,1},\ldots,x_{1,p})(x_{2,1},\ldots,x_{2,p})^{\pi_1^{-1}}\cdots(x_{t,1},\ldots,x_{t,p})^{\pi_{t-1}^{-1}}\pi_t \\ &= (x_{1,1},\ldots,x_{1,p})(x_{2,1\pi_1},\ldots,x_{2,p\pi_1})\cdots(x_{t,1\pi_{t-1}},\ldots,x_{t,p\pi_{t-1}})\pi_t \\ &= (x_{1,1}x_{2,1\pi_1}\cdots x_{t,1\pi_{t-1}}\,,\ \ldots\ ,\ x_{1,p}x_{2,p\pi_1}\cdots x_{t,p\pi_{t-1}})\pi_t\end{aligned}$$

Finally, the statement about the length of the characteristics follows from the corresponding statements in the proof of Theorem 5.2, by noting that $m_1m_{1\pi}\cdots m_{1\pi^{p-1}}$ is the first component of $((m_1,m_2,\ldots,m_p)\pi)^p$. This proof of the lemma is complete.

6.4 Normal Symbols

For the flat fake-$E_{f,p}$-symbol Δ, define

$$FPS(\Delta) = \left\{\mathcal{O} \in PS(\Delta) \;\middle|\; \left|\overline{\chi(\mathcal{O})}\right| < \infty\right\} \text{ and}$$

$$IPS(\Delta) = \left\{\mathcal{O} \in PS(\Delta) \;\middle|\; \left|\overline{\chi(\mathcal{O})}\right| = \infty\right\}.$$

These two sets are effectively computable from Δ, by Theorem 5.2. We call Δ a *normal* symbol for g, if it is a $\langle g\rangle$-fake-$E_{f,p}$-symbol and, in addition, satisfies the following condition

(N) For every expansion Γ of Δ which is also a flat fake-$E_{f,p}$-symbol and for all $\mathcal{O} \in IPS(\Gamma)$, $\mathcal{O}$ is of class II.

Note that (N) is also a condition for Δ itself since Δ is trivially an expansion of Δ.

The reason for this definition is that we would like to write a flat fake-$E_{f,p}$-symbol Δ for g as the combination of two flat fake-$E_{f,p}$-symbols Δ_F, Δ_I; the former containing only pseudo-orbits with finite characteristic, and the latter having only pseudo-orbits with infinite characteristics and trivial columns. This decomposition will define two elements g_F and g_I in a natural way, the only trouble is that this definition is not independent of the choice of the flat fake-$E_{f,p}$-symbol. As a pseudo-orbit with infinite characteristic can lead to a pseudo-orbit with finite characteristic under pseudo-orbit expansions (see the example below). We show in Lemma 6.3 that for a normal symbol for g this will be well defined. Let us first give an example of a flat fake-$E_{f,p}$-symbol which is not normal.

Example. Consider the group $\mathcal{H}_{2,2}$ and let ω, σ, and τ be the 2-periodic sequences with periods 11, 10, and 01, respectively. Then it is an easy exercise application of Theorem 5.2 that γb_ω and $b_\tau\gamma$ have infinite respectively finite order. Suppose

$$\mathcal{O} = \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ b_\sigma & \gamma & b_\omega & \gamma \\ o_2 & o_3 & o_4 & o_1 \end{pmatrix}$$

is a pseudo orbit of a flat fake-$E_{f,p}$-symbol Δ for some element of $\mathcal{H}_{2,2}$. Since $(b_\sigma\gamma b_\omega\gamma)^\Phi = (\gamma b_\omega, b_\tau\gamma)$, we have that $\mathcal{O}$ of class I and hence Δ is not normal. Note that $\mathcal{O}$ has infinite characteristic. By Lemma 6.2 and the usual expansion rules, in the simple pseudo-orbit expansion of Δ at $\mathcal{O}$, $\mathcal{O}$ leads to

$$\mathcal{O}_1 = \begin{pmatrix} o_1a_1 & o_2a_1 & o_3a_2 & o_4a_2 \\ \gamma & 1 & b_\omega & 1 \\ o_2a_1 & o_3a_2 & o_4a_2 & o_1a_1 \end{pmatrix} \text{ and } \mathcal{O}_2 = \begin{pmatrix} o_1a_2 & o_2a_2 & o_3a_1 & o_4a_1 \\ b_\tau & 1 & \gamma & 1 \\ o_2a_2 & o_3a_1 & o_4a_1 & o_1a_2 \end{pmatrix}.$$

Observe that $\mathcal{O}_1$ has infinite characteristic, whereas $\mathcal{O}_2$ has finite characteristic.

The following lemma says that for a normal symbol the pathology of the above example cannot occur.

***Lemma* 6.3** *Let Δ be a normal symbol for $g \in \mathcal{H}_{f,p}$, and let Γ be a pseudo-orbit expansion of Δ. Then the following hold.*

a) Γ is a normal symbol for g

b) If $\mathcal{U} \in FPS(\Gamma)$*, then* $\mathrm{top}(\mathcal{U}) \subset \mathrm{top}(\mathcal{O})W_p$ *for some* $\mathcal{O} \in FSP(\Delta)$.

Proof. To see a) it suffices to note that Γ is a $\langle g\rangle$-fake-$E_{f,p}$-symbol satisfying (N), since all its pseudo-orbit expansions are also pseudo-orbit expansions of Δ. To prove b) it suffices to consider simple pseudo-orbit expansions of Δ. By case 2 of Lemma 6.2, a pseudo-orbit of class II with infinite characteristic leads to precisely one pseudo-orbit with infinite characteristic under a simple pseudo-orbit expansion, and condition (N) forces this pseudo-orbit to be of class II again. so $\mathcal{U}$ cannot be the expansion of a pseudo-orbit of class II with infinite characteristic, and the lemma is proved. The existence of normal symbols is our next task.

***Proposition* 6.4** *Suppose* $g \in \mathcal{H}_{f,p}$ *has a flat fake-*$E_{f,p}$*-symbol. Then* g *has a normal symbol.*

Proof. Let us start with a verbal version of Lemma 6.3 b): for every pseudo-orbit expansion of a normal symbol, pseudo-orbits with finite characteristic can only come from pseudo-orbits with finite characteristic. From the proof of that statement it is clear that only pseudo-orbits of class I with infinite characteristic may lead to 'new' pseudo-orbits with finite characteristic. In this case, the length of the characteristic of the new pseudo-orbits is strictly smaller than the length of the original characteristic, by Lemma 6.2. Therefore, new pseudo-orbits with finite characteristic, in the sense that they come from a pseudo-orbits with infinite characteristic, cannot occur infinitely many times in the process of performing simple pseudo-orbit expansions at pseudo-orbits with infinite characteristic over and over again. We will know when to stop, when we find that two expansions of the same pseudo-orbit with infinite characteristic have equal characteristics, which must happen by the arguments in the proof of Theorem 5.2. The proof is complete.

6.5 THE CONJUGACY PROBLEM: I

Having normal symbols at hand, we show here that we can divide the conjugacy problem for elements with flat symbols into two parts: those with pseudo-orbits of finite characteristics on the one hand, and those with either trivial columns or pseudo-orbits of infinite characteristics on the other hand. The latter part is easier and proven to be solvable also in this section. First we need to introduce more notation.

Let Δ be a normal symbol for g. Define

$$\mathrm{top}(\Delta)_I = \{\mathrm{top}(\mathcal{C}) \mid \mathcal{C} \in \mathcal{O} \in IPS(\Delta)\},$$

$$\mathrm{top}(\Delta)_F = \{\mathrm{top}(\mathcal{C}) \mid \mathcal{C} \in \mathcal{O} \in FPS(\Delta)\},$$

$$\Delta_I = (\bigcup_{\mathcal{O} \in IPS(\Delta)} \mathcal{O}) \cup \left\{ \begin{pmatrix} b \\ 1 \\ b \end{pmatrix} \mid b \in \mathrm{top}(\Delta)_F \right\}$$

and

$$\Delta_F = (\bigcup_{\mathcal{O} \in FPS(\Delta)} \mathcal{O}) \cup \left\{ \begin{pmatrix} b \\ 1 \\ b \end{pmatrix} \mid b \in \mathrm{top}(\Delta)_I \right\}.$$

Remark. Note that $\mathrm{top}(\Delta)_I$ and $\mathrm{top}(\Delta)_F$ are disjoint with union $\mathrm{top}(\Delta)$, and that $\mathrm{bot}(\{\mathcal{C} \mid \mathrm{top}(\mathcal{C}) \in \mathrm{top}(\Delta_X)\}) = \mathrm{top}(\Delta_X)$ for $X = F, I$. Hence Δ_I and Δ_F are well defined. It should be clear that Δ_F defines a periodic element, whereas Δ_I defines an element of infinite order.

***Lemma* 6.5** *Let Δ be a normal symbol for $g \in \mathcal{H}_{f,p}$ and let g_F and g_I be the elements defined by Δ_F and Δ_I, respectively. Then g_F and g_I are independent of the choice of the normal symbol Δ and $g = g_I g_F = g_F g_I$.*

Proof. A similar proof to that for Lemma 6.1 shows that there is a unique minimal normal symbol Δ for g such that every other normal symbol for g is a pseudo-orbit expansion of Δ. In fact every pseudo-orbit expansion Γ of Δ is a normal symbol for g, by Lemma 6.3, which also shows that $\mathrm{top}(\Gamma)_F \subset \mathrm{top}(\Delta)_F W_p$. Together with $\mathrm{top}(\Gamma)_I \subset \mathrm{top}(\Delta)_I W_p$ which obviously holds, we get $\Gamma_F \succeq \Delta_F$ and $\Gamma_I \succeq \Delta_I$. Hence Γ_F is a symbol for g_F and Γ_I is a symbol for g_I, whence the definition of g_F and g_I are independent of the choice of normal symbol Δ for g. It is plain from the definitions that $g = g_I g_F = g_F g_I$, and the lemma is proved.

***Proposition* 6.6** *Assume $g, h \in \mathcal{H}_{f,p}$ have flat fake-$E_{f,p}$-symbols. Then g and h are conjugate in $\mathcal{H}_{f,p}$ if and only if g_F is conjugate to h_F and g_I is conjugate to h_I.*

Proof. First suppose that, for some $x \in \mathcal{H}_{f,p}$, $x^{-1}gx = h$ is a relation in $\mathcal{H}_{f,p}$. Now g and h have normal symbols, by Lemma 6.4, and, by Lemma 6.3 a), we can expand (6.2) so that $\Sigma^{-1}\Delta\Sigma = \Gamma$ holds, where Δ and Γ are normal symbols and Σ is a fake-$E_{f,p}$-symbol. It follows by what was said about the characteristics after (6.3) that $\Sigma^{-1}\Delta_F\Sigma = \Gamma_F$ and $\Sigma^{-1}\Delta_I\Sigma = \Gamma_I$. Thus g_F is conjugate to h_F and g_I is conjugate to h_I, by Lemma 6.5.

For the converse assume that $x^{-1}g_F x = h_F$ and $y^{-1}g_I y = h_I$ hold in $\mathcal{H}_{f,p}$ for some $x, y \in \mathcal{H}_{f,p}$. Then there are fake-$E_{f,p}$-symbols Σ, Λ, Δ, and Γ for x, y, g, and h, respectively, such that

$$\Sigma^{-1}\Delta_F\Sigma = \Gamma_F \qquad \text{and} \qquad \Lambda^{-1}\Delta_I\Lambda = \Gamma_I.$$

Define the essential parts of Σ and Λ to be the following collections of columns

$$\Sigma_e = \{\mathcal{C} \in \Sigma \,|\, \mathrm{top}(\mathcal{C}) \in \mathrm{top}(\Delta)_F\}$$

and

$$\Lambda_e = \{\mathcal{C} \in \Lambda \,|\, \mathrm{top}(\mathcal{C}) \in \mathrm{top}(\Gamma)_F\}.$$

By the remark preceding Lemma 6.5, $\mathrm{top}(\Sigma_e)$ and $\mathrm{top}(\Lambda_e)$ are disjoint, $\mathrm{top}(\Sigma_e) \cup \mathrm{top}(\Lambda_e) = \mathrm{top}(\Delta)$, $\mathrm{top}(\Sigma_e) = \mathrm{bot}(\Sigma_e)$, and $\mathrm{top}(\Lambda_e) = \mathrm{bot}(\Lambda_e)$. Hence $\Omega = \Sigma_e \cup \Lambda_e$ is a fake-$E_{f,p}$-symbol and it is easy to check that the combination $\Omega^{-1}\Delta\Omega$ exists and is equal to Γ. This proves the Proposition.

We end this section by showing that we can decide whether g_I and h_I are conjugate. In the following sections we tackle the considerably more difficult task to decide whether g_F and h_F are conjugate. One further definition is required. For $a, b, r \in \mathbb{N}$, we write $a \equiv^* b \pmod r$, if $a \equiv b \pmod r$ and $a = 0$ if and only if $b = 0$.

***Proposition* 6.7** *Let g and h be elements of $\mathcal{H}_{f,p}$ which have flat fake-$E_{f,p}$-symbols. Then it is decidable whether g_I and h_I are conjugate.*

Proof. First recall the definition of the trace of a pseudo-orbit (p. 73). For a flat fake-$E_{f,p}$-symbol Σ, let $tt(\Sigma)$ be the number of pseudo-orbits of Σ with trace zero (*t*rivial *t*race) and $nt(\Sigma)$ the number of pseudo-orbits of Σ with trace one. Suppose g_I and h_I are defined by Δ_I respectively Γ_I, where Δ and Γ are normal symbols for g and h. Assume for a moment that Σ is a pseudo-orbit expansion of Δ_I. Then $nt(\Sigma) = nt(\Delta_I)$ and $tt(\Sigma) \equiv^* tt(\Delta_I) \pmod{p-1}$, by Lemma 6.2 and condition (N). Hence we see from the discussion in Section 6.2 that the following three conditions are necessary for g_I and h_I to be conjugate.

1) $nt(\Delta_I) = nt(\Gamma_I)$
2) $tt(\Delta_I) \equiv^* tt(\Gamma_I) \pmod{p-1}$
3) There are suitably ordered pseudo-orbit expansions Δ_I' and Γ_I' of Δ_i respectively Γ_I and elements $y_1, \ldots, y_r \in E_{f,p} \cup C$ such that, for suitable listings of their pseudo-orbits $\mathcal{O}_1, \ldots, \mathcal{O}_r$ respectively $\mathcal{U}_1, \ldots, \mathcal{U}_r$, $\chi(\mathcal{O}_i)^{y_i} = \chi(\mathcal{U})$ for $1 \le i \le r$, where $r = nt(\Delta)$.

That these conditions are also sufficient is a consequence of the first part of Lemma 6.2, which together with condition 2) implies that we can choose Δ_I' and Γ_I' with the same number of trivial columns, and the fact that every fake-$E_{f,p}$-symbol defines an element of $\mathcal{H}_{f,p}$. Let us show how to check in finite time whether these three conditions are satisfied by the given Δ_I and Γ_I. Clearly, we only need to check condition 3) in case 1) and 2) hold already.

We claim that there is a finite subset S of M so that for every pseudo-orbit $\mathcal{U}$ of any pseudo-orbit expansion Σ of Δ_I, $\chi(\mathcal{U}) \subset S$. This follows from the last statement of Lemma 6.2 by which the length of all possible characteristic of such $\mathcal{U}$ are bounded. Now we are left to check finitely many things to find whether condition 3) can be satisfied, since there are only finitely many pseudo-orbit each with only finitely many possible characteristics for its expansions, which can be cyclically reordered in finitely many ways, and only finitely many elements to choose the y_i from. The proof is complete.

Remark. It is worth noting that until now we have not used our assumption that p is a prime.

6.6 Legal Symbols and their Index

Now we deal with the question whether g_F and h_F are conjugate for given elements g and h of $\mathcal{H}_{f,p}$ which admit flat fake-$E_{f,p}$-symbols. In other words we attack the conjugacy problem for elements of finite order which, by Proposition 5.7, are precisely those elements with a flat $E_{f,p}$-symbol.

Call Δ is a *legal symbol* for g if it is a $\langle g\rangle$-fake-$E_{f,p}$-symbol and an expansion of a flat $E_{f,p}$-symbol for g. In particular, an element of $\mathcal{H}_{f,p}$ has a legal symbol if and only if it has finite order. The following result divides the pseudo-orbits of legal symbols into two genuinely different classes. Recall that $C = \langle\gamma\rangle$, where γ has the symbol (5.1).

***Lemma* 6.8** *Suppose Γ is a legal symbol for $g \in \mathcal{H}_{f,p}$. Then for each pseudo-orbit $\mathcal{O}$ of g in Γ one (and only one) of the following holds.*

I) *All middle row entries of $\mathcal{O}$ belong to C.*

II) *One of the middle row entries of $\mathcal{O}$ belongs to $E_{f,p} \setminus \{1\}$. Furthermore, in this case all middle row entries of $\mathcal{O}$ are in $E_{f,p}$. Consequently, the characteristic of $\mathcal{O}$ is independent of the order of the columns of Γ.*

We will speak of pseudo-orbits of *type* I or II depending on which statement of the lemma they satisfy and we denote the type of $\mathcal{O}$ by $t(\mathcal{O})$.

Proof. First of all, note that every flat $E_{f,p}$-symbols satisfies the dichotomy of the lemma. As Γ is a pseudo-orbit expansion of some $E_{f,p}$-symbol, it suffices to check that a simple pseudo-orbit expansion of a flat fake-$E_{f,p}$-symbol that satisfies the lemma also satisfies the lemma. This can be seen from Lemma 6.2 as follows. If a pseudo-orbit is of type I, then all the $x_{i,j}$ in the proof of that lemma are trivial, so all the resulting pseudo-orbits are of type I again. If, on the other hand, a pseudo-orbit is of type II, then it is of class I, and all the π_i are trivial. It follows from (4.6) that only middle row entries from either $E_{f,p}$ or C contribute to the characteristic of resulting pseudo-orbit. The final statement follows because $E_{f,p}$ is abelian, by Lemma 4.9, and the lemma is proved.

It is convenient to introduce the following terminology for the legal symbol Δ. The *rank of* Δ, denoted $r(\Delta)$, is given by

$$r(\Delta) = |PS(\Delta)|.$$

For $\mathcal{O} \in PS(\Delta)$, we define the *index* $i(\mathcal{O})$ *of* $\mathcal{O}$ by

$$i(\mathcal{O}) = (\chi(\mathcal{O}), |\mathcal{O}|),$$

and we call the unordered tuple

$$\{i(\mathcal{O})\}_{\mathcal{O} \in PS(\Delta)}$$

the *index of* Δ which we denote by $i(\Delta)$. A closer inspection of the proof of Lemma 6.8 easily gives the following result.

***Proposition* 6.9** *Let Δ be a legal symbol for g. If Γ is the result of the simple pseudo-orbit expansion of Δ at $\mathcal{O} \in PS(\Delta)$, then the following hold.*

a) $r(\Gamma) = \begin{cases} r(\Delta), & \text{if } 1 \neq \overline{\chi(\mathcal{O})} \in C \\ r(\Delta) + p - 1, & \text{otherwise} \end{cases}$

b) $i(\Gamma)$ is obtained from $i(\Delta)$ by deleting $i(\mathcal{O})$ and adding

(i) $(1, cp)$, *if* $i(\mathcal{O}) = (\delta, c)$ *with* $1 \neq \delta \in C$, *or*

(ii) p *times* $(1, c)$, *if* $i(\mathcal{O}_i) = (1, c)$, *or*

(iii) $p - 2$ *times* $(1, c)$, (δ, c) *and* (e', c), *in case* $i(\mathcal{O}) = (e, c)$, $1 \neq e$, *and* e *has the symbol (5.2).*

In particular, the number of pseudo-orbits of g of type II *with non-trivial characteristic is independent of the chosen legal symbol for g.*

The index of a legal symbol allows a first necessary and sufficient criterion for period elements to be conjugate.

***Proposition* 6.10** *Let $g, h \in \mathcal{H}_{f,p}$ be elements of finite order. Then g and h are conjugate in $\mathcal{H}_{f,p}$ if and only if there are legal symbols Δ and Γ for g respectively h, such that $i(\Delta) = i(\Gamma)$, i.e. there is a bijection $\alpha : PS(\Delta) \longrightarrow PS(\Gamma)$ with $i(\mathcal{O}^\alpha) = i(\mathcal{O})$.*

Proof. First assume $x^{-1}gx = h$ is a relation in $\mathcal{H}_{f,p}$. Doing a few expansions we may assume that we are in the situation of Section 6.2 with Δ and Γ of (6.3) being a flat $E_{f,p}$-symbol and a legal symbol, respectively. Obviously $|\mathcal{U}| = |\mathcal{O}|$, so once we establish $\chi(\mathcal{O}) = \chi(\mathcal{U})$, Δ and Γ clearly have the same index. By Lemma 6.8 and the form of Δ and Γ, we know $\chi(\mathcal{O}) \in E_{f,p}$, $\chi(\mathcal{U}) \in C \cup E_{f,p}$ and that $\chi(\mathcal{U})$ is independent of the order of multiplication. But from (6.4), $\chi(\mathcal{U}) = \chi(\mathcal{O})^{y_i}$ for $0 \leq i \leq t-1$. Surely, we may now assume $\chi(\mathcal{O}) \neq 1$. From the definition of C and $E_{f,p}$ we see that $y_i \in E_{f,p}$ for $0 \leq i \leq t-1$. Since $E_{f,p}$ is abelian we get $\chi(\mathcal{O}) = \chi(\mathcal{U})$, as required.

For the converse, let Δ and Γ be legal symbols for g and h, respectively, with $i(\Delta) = i(\Gamma)$. For $1 \leq j \leq r(\Delta)$, let $\mathcal{O}_j$ and $\mathcal{O}_j^\alpha$ be given by

$$\begin{pmatrix} o_{1,j} & \cdots & o_{t_j-1,j} & o_{t_j,j} \\ x_{1,j} & \cdots & x_{t_j-1,j} & x_{t_j,j} \\ o_{2,j} & \cdots & o_{t_j,j} & o_{1,j} \end{pmatrix} \text{ and } \begin{pmatrix} w_{1,j} & \cdots & w_{t_j-1,j} & w_{t_j,j} \\ z_{1,j} & \cdots & z_{t_j-1,j} & z_{t_j,j} \\ w_{2,j} & \cdots & w_{t_j,j} & w_{1,j} \end{pmatrix}$$

respectively. Now define for $1 \leq j \leq r(\Delta)$ the following elements of M inductively, $y_{1,j} = 1$ and $y_{i+1,j} = x_{i,j}^{-1} y_{i,j} z_{i,j}$ for $1 \leq i \leq t_j - 1$. Finally, let $k \in \mathcal{H}_{f,p}$ be the element with M-symbol

$$\Phi = \bigcup_{j=1}^{r(\Delta)} \left\{ \begin{pmatrix} o_{i,j} \\ y_{i,j} \\ w_{i,j} \end{pmatrix} \;\middle|\; 1 \leq i \leq t_j \right\}.$$

(Observe that every M-symbol defines an element of $\mathcal{H}_{f,p}$.) Having in mind that $x_{1,j} \cdots x_{t_j,j} = \chi(\mathcal{O}_|) = \chi(\mathcal{O}_|{}^\alpha) = z_{1,j} \cdots z_{t_j,j}$, it is now straightforward to check that $\Phi^{-1}\Delta\Phi$ exists and is equal to Γ. This completes the proof of the Proposition.

6.7 The Conjugacy Problem: II

Before we turn to the main result of this section we need yet more notation and describe some reductions. Let Δ be a legal $\langle g \rangle$-fake-$E_{f,p}$-symbol, then

$$|g| = \begin{cases} p|\pi_g(\Delta)|, & \text{if } \Delta^{\pi_g(\Delta)} \text{ has pseudo-orbits with trace one} \\ |\pi_g(\Delta)|, & \text{otherwise.} \end{cases} \tag{6.5}$$

Remark. This is the first time we use that p is a prime.

For each divisor d of the order of g we define

$$
\begin{aligned}
&N_d(\Delta) = \{\mathcal{O} \in PS(\Delta) \,|\, i(\mathcal{O}) = (1,d)\},\\
&F_d(\Delta) = \{\mathcal{O} \in PS(\Delta) \,|\, i(\mathcal{O} = (\delta,d), 1 \neq \delta \in C\},\\
&M_d(\Delta) = \{\mathcal{O} \in PS(\Delta) \,|\, i(\mathcal{O}) = (e,d), 1 \neq e \in E_{f,p}\},\\
&n_d(\Delta) = \begin{cases} |N_d(\Delta)| + |F_{d/p}(\Delta)|, & \text{if } p \text{ divides } d\\ |N_d(\Delta)|, & \text{otherwise,} \end{cases}\\
&m_d(\Delta) = |M_d(\Delta)|, \text{ and finally}\\
&d - type(\Delta) = \{\chi(\mathcal{O})\}_{\mathcal{O} \in M_d(\Delta)},
\end{aligned}
$$

by which we mean an unordered $m_d(\Delta)$-tuple. We chose this notation because we distinguish *n*ormalised and *f*ake pseudo-orbits of type I and pseudo-orbits with non-trivial characteristic in the group $E_{f,p} \subset M$. Let us call $\hat{\Delta}$ a *legal expansion of* Δ, if $\hat{\Delta}$ is a legal symbol and also an expansion of Δ.

Assume now that we are given $g, h \in \mathcal{H}_{f,p}$ of finite order. First we find $\langle g\rangle$- and $\langle h\rangle$-$E_{f,p}$-symbols Δ and Γ, respectively. Note that this can be done effectively (see proof of Proposition 5.7). By Proposition 6.10, it suffices to give an effective procedure which decides whether there are Δ' and Γ', legal $\langle g\rangle$- and $\langle h\rangle$-fake-$E_{f,p}$-symbols, respectively, such that $i(\Delta') = i(\Gamma')$. In addition, we may assume that Δ' and Γ' expand Δ and Γ, so that all possible indices of such Δ' and Γ' can be effectively computed from those of Δ and Γ, by Proposition 6.9.

The following conditions are clearly necessary because they are invariant under simple pseudo-orbit expansion, by Proposition 6.9.

1) $|g| = |h|$
2) $r(\Delta) \equiv r(\Gamma) \pmod{p-1}$
3) $m_d(\Delta) = m_d(\Gamma)$ for all divisors d of $|g|$

These conditions can obviously be checked. Furthermore, since $E_{f,p}$ is finite, there are only finitely many possibilities for the $d - type$ of a legal expansion of Δ, and similarly for Γ. Therefore, by Proposition 6.9 b) (iii), we can also check if the following condition can be met.

4) There exist legal expansions of Δ and Γ with the same $d - type$ for all divisors d of $|g|$.

Recall that $a \equiv^* b \pmod{r}$ if $a \equiv b \pmod{r}$ and $a = 0$ if and only if $b = 0$. We proceed with the following weakening of Proposition 6.10.

***Lemma* 6.11** *Suppose* Δ *and* Γ *satisfy the requirements* $1) - 4)$. *Then* g *and* h *are conjugate in* $\mathcal{H}_{f,p}$ *if and only if*

5) $n_d(\Delta) \equiv^* n_d(\Gamma) \pmod{p-1}$ *for all divisors* d *of* $|g|$.

Proof. *Only if* follows directly from Proposition 6.10. We prove the *if* direction. By Proposition 6.9 b) (ii), it is clear that Δ and Γ have legal expansions Δ' and Γ' with $n_d(\Delta') = n_d(\Gamma')$ for all divisors d of $|g|$. From Proposition 6.9 b) (i) and the definition of n_d we see that we can expand Δ' and Γ' further to obtain $\hat{\Delta}$ and $\hat{\Gamma}$ with $n_d(\hat{\Delta}) = n_d(\hat{\Gamma})$ and $F_d(\hat{\Delta}) = F_d(\hat{\Gamma}) = \emptyset$ for all divisors d of $|g|$. Since all these expansions leave the $d-type$ fixed, it follows that $i(\hat{\Delta}) = i(\hat{\Gamma})$. Now, by Proposition 6.10, g and h are conjugate, which completes the proof of the lemma.

The difficulty we have to overcome now is that pseudo-orbit expansions of orbits of type II can and will produce new fake pseudo-orbits and therefore may change the residue class of some n_d. So that we cannot be sure that g and h are not conjugate if the condition in the lemma is not satisfied for some choice of legal symbols satisfying $1) - 4)$. Hence we need a closer investigation of this phenomenon.

From now on will denote elements of $E_{f,p}$ by their defining sequence, i.e., $e = b_\omega \in E_{f,p}$ is denoted by ω. And for ω, let $P(\omega)$ denote the period of ω; thus, $P(\omega)$ is the least non-zero positive integer with $\omega^{\kappa^{P(\omega)}} = \omega$, where κ is the 'forget' (4.4). Furthermore, we define the *multiplicator* $M(\omega)$ of ω by

$$M(\omega) = |\{\omega(i) \mid \omega(i) \neq 0,\ 0 \leq i < P(\omega)\}|.$$

***Theorem* 6.12** *Suppose* $g, h \in \mathcal{H}_{f,p}$ *have finite order. Then it is effectively decidable if* g *and* h *are conjugate in* $\mathcal{H}_{f,p}$.

Proof. Let Δ and Γ be legal symbols for g respectively h such that $1) - 4)$ hold. We are going to describe an effective procedure which decides whether Δ and Γ have legal expansions satisfying 5).

Suppose for a moment that $d - type(\Delta) = (\omega_1, \ldots, \omega_{m_d(\Delta)})$ and let $M_i = M(\omega_i)$. Then it is an easy consequence of Proposition 6.9 b) (iii) that there is a legal expansion Δ' of Δ such that $d - type(\Delta) = d - type(\Delta')$ and

$$n_l(\Delta') = \begin{cases} n_d(\Delta) + (p-2)\sum_{i=1}^{m_d(\Delta)} \lambda_i M_i, & \text{if } l = d \\ n_{pd}(\Delta) + \sum_{i=1}^{m_d(\Delta)} \lambda_i M_i, & \text{if } l = pd \\ n_l(\Delta), & \text{if } l \neq d, pd \end{cases},$$

for any choice of $\lambda_i \in \mathbb{N}$.

Let $D(g)$ denote the set of all divisors of $|g|$ that are not divisible by p and let $\eta \in \mathbb{N}$ be such that p^η is the highest power of p dividing $|g|$. Hence, if

$d' \in D(g)$ then every legal expansion $\hat{\Delta}$ of Δ with $d'-type(\hat{\Delta}) = d'-type(\Delta)$ satisfies

$$n_{d'}(\hat{\Delta}) = n_{d'}(\Delta) + (p-2) \sum_{i=1}^{m_{d'}(\Delta)} \lambda_i M_i$$

for suitable $\lambda_i \in \mathbb{N}$. Furthermore, since Δ and Γ have the same $d'-type$, there are legal expansions Δ' and Γ' with the same $d'-type$ and $n_{d'}(\Delta') \equiv^* n_{d'}(\Gamma') \pmod{p-1}$ if and only if

$$n_{d'}(\Delta) - n_{d'}(\Gamma) \equiv \sum_{i=1}^{m_{d'}(\Delta)} \xi_i M_i \pmod{p-1} \tag{6.6}$$

has a solution with $\xi_i \in \mathbb{Z}$. This is because we can alter Δ and Γ and the fact that for Γ exactly the same M_i are to be used.

We may now assume that (6.6) has a solution for all $d' \in D(g)$. It is clear that for legal expansions Δ' and Γ' with $d'-type(\Delta') = d'-type(\Gamma')$ and $n'_d(\Delta') \equiv^* n'_d(\Gamma') \pmod{p-1}$, $d' \in D(g)$, we have new values for $n_{d'p}(\Delta')$ and $n_{d'p}(\Gamma')$ in comparison to Δ and Γ. But, by repeating the argument above for divisors of the form $d'p^i$, $d' \in D(g)$, $1 \leq i \leq \eta - 1$ in increasing order, and noting that expansions of pseudo-orbits of index $(e, d'p^i)$ $(1 \neq e \in E_{f,p})$ leave $n_{d'p^j}$ unchanged for $0 \leq j < i$, it is decidable whether g and h have legal symbols Δ' and Γ' satisfying $n_d(\Delta') \equiv^* n_d(\Gamma') \pmod{p-1}$ for all divisors d of $\frac{|g|}{p}$ or $|g|$ according to whether $|g|$ is divisible by p or not. Thus, if for such Δ' and Γ' also $n_d(\Delta') \equiv^* n_d(\Gamma') \pmod{p-1}$ holds for all other divisors d of $|g|$, then g and h are conjugate in $\mathcal{H}_{f,p}$, by Lemma 6.11. On the other hand it is clear from the procedure that, if this is not the case then g and h do not have legal symbols which satisfy 5), whnece g and h cannot be conjugate in $\mathcal{H}_{f,p}$, by Proposition 6.10. The proof of Theorem 6.12 is now complete.

Combining this result with Lemma 5.4, Proposition 6.6 and Proposition 6.7 we have

***Theorem* 6.13** *The conjugacy problem for elements of $G_{f,p}$ with flat M-symbols is solvable if p is a prime.*

We would like to point out that this is the obvious adoption of the method used in [24] to classify conjugacy classes of finite subgroups of $G_{n,r}$. Only, here it gets a little more complex, as we have to take the group M into account. Let us conclude this thesis by showing that there are as many isomorphism classes among the $\mathcal{H}_{f,p}{}'$ as possible and a proof of Theorem A stated in the introduction.

***Theorem* 6.14** *If p and q are primes and $p \neq q$, then $\mathcal{H}_{f,p}$ is not isomorphic to $\mathcal{H}_{f,q}$ and $\mathcal{H}_{f,p}{}'$ is not isomorphic to $\mathcal{H}_{f,q}{}'$*

Proof. First of all, the group $\mathcal{H}_{f,p}$ has infinitely many conjugacy classes of elements of order p because there are symbols Δ whose columns have equal top and bottom row entries with $m_d(\Delta) = k$ for every $k \in \mathbb{N}$. On the other hand, it has only finitely many conjugacy classes of elements of order q, since such a conjugacy class is characterised by the numbers of fixed points and q-cycles both modulo $p-1$. The same arguments goes through for the derived subgroups. This completes the proof.

Proof of Theorem A. Put Theorems 4.10, 4.14 and 6.14 together. The proof and thesis are complete.

Appendix A

Some Calculations

Here we present detailed calculations of relations used in the proof of Lemma 3.9. Since we are dealing with elements of $G_{p,1}$ which all have 1-symbols we do not print the ones in the middle rows.

First we show how to obtain the relation $\epsilon = \sigma_{x_{j_1}}^{\delta_{j_1}} \cdots \sigma_{x_{j_r}}^{\delta_{j_r}} \delta \sigma_\pi$ used in the first part of the proof of Lemma 3.9. We omit the columns

$$\begin{pmatrix} a_n a_j \\ 1 \\ a_n a_j \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_i \\ 1 \\ a_i \end{pmatrix}$$

for $2 \leq j \leq n$ and $2 \leq i \leq n-1$ as they are not altered by the elements under consideration. In the following $1 \leq i \leq n$.

$$\left.\begin{array}{cccccccl}
a_n a_1 & a_1 a_{k_1 \pi} & \cdots & a_1 a_{k_t \pi} & a_1 a_{j_1 \pi} a_{i x_{i_{j_1}}} & \cdots & a_1 a_{j_r \pi} a_{i x_{i_{j_r}}} & \\
 & & & & & & & \downarrow \epsilon^{-1} \\
a_1 & a_n a_1 a_{k_1} & \cdots & a_n a_1 a_{k_t} & a_n a_1 a_{j_1} a_i & \cdots & a_n a_1 a_{j_r} a_i & \\
 & & & & & & & \downarrow \sigma_{x_{j_1}}^{\delta_{j_1}} \\
a_1 & a_n a_1 a_{k_1} & \cdots & a_n a_1 a_{k_t} & a_n a_1 a_{j_1} a_{x_{j_1}} & \cdots & a_n a_1 a_{j_r} a_i & \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \\
 & & & & & & & \downarrow \sigma_{x_{j_r}}^{\delta_{j_r}} \\
a_1 & a_n a_1 a_{k_1} & \cdots & a_n a_1 a_{k_t} & a_n a_1 a_{j_1} a_{x_{j_1}} & \cdots & a_n a_1 a_{j_r} a_{i x_{j_r}} & \\
 & & & & & & & \downarrow \delta \\
a_n a_1 & a_1 a_{k_1} & \cdots & a_1 a_{k_t} & a_1 a_{j_1} a_{i x_{i_{j_1}}} & \cdots & a_1 a_{j_r} a_{i x_{i_{j_r}}} & \\
 & & & & & & & \downarrow \sigma_\pi \\
a_n a_1 & a_1 a_{k_1 \pi} & \cdots & a_1 a_{k_t \pi} & a_1 a_{j_1 \pi} a_{i x_{i_{j_1}}} & \cdots & a_1 a_{j_r \pi} a_{i x_{i_{j_r}}} &
\end{array}\right) 1$$

Computing the relation $\alpha \delta_1 \delta \sigma_\pi \delta \delta_1 \alpha^{-1} \tau^{-1} = \epsilon$ in the case $m = p$. Assume that $a_i\pi = a_1$, that is $i\pi = 1$.

$$\left.\begin{array}{ccccccccccl}
a_1 & w_2 & \cdots & w_i & \cdots & w_p & w_{p+1} & \cdots & w_{s-1} & w_s & \\
 & & & & & & & & & & \downarrow \alpha \\
a_1 & a_pa_1a_2 & \cdots & a_pa_1a_i & \cdots & a_pa_1a_p & z_{p+1} & \cdots & z_{s-1} & a_pa_1a_1 & \\
 & & & & & & & & & & \downarrow \delta_1 \\
a_pa_1a_1 & a_pa_1a_2 & \cdots & a_pa_1a_i & \cdots & a_pa_1a_p & z_{p+1} & \cdots & z_{s-1} & a_1 & \\
 & & & & & & & & & & \downarrow \delta \\
a_1a_1 & a_1a_2 & \cdots & a_1a_i & \cdots & a_1a_p & z_{p+1} & \cdots & z_{s-1} & a_pa_1 & \\
 & & & & & & & & & & \downarrow \sigma_\pi \\
a_1(a_1\pi) & a_1(a_2\pi) & \cdots & a_1a_1 & \cdots & a_1(a_p\pi) & z_{p+1} & \cdots & z_{s-1} & a_pa_1 & \\
 & & & & & & & & & & \downarrow \delta \\
a_pa_1(a_1\pi) & a_pa_1(a_2\pi) & \cdots & a_pa_1a_1 & \cdots & a_pa_1(a_p\pi) & z_{p+1} & \cdots & z_{s-1} & a_1 & \\
 & & & & & & & & & & \downarrow \delta_1 \\
a_pa_1(a_1\pi) & a_pa_1(a_2\pi) & \cdots & a_1 & \cdots & a_pa_1(a_p\pi) & z_{p+1} & \cdots & z_{s-1} & a_pa_1a_1 & \\
 & & & & & & & & & & \downarrow \alpha^{-1} \\
w_{1\pi} & w_{2\pi} & \cdots & a_1 & \cdots & w_{p\pi} & w_{p+1} & \cdots & w_{s-1} & w_s & \\
 & & & & & & & & & & \downarrow \tau^{-1} \\
a_1(a_1\pi) & a_1(a_2\pi) & \cdots & a_1a_1 & \cdots & a_1(a_p\pi) & x_{p+1} & \cdots & x_{s-1} & x_s &
\end{array}\right)\epsilon$$

Computing the relation $\tau''\eta_2\cdots\eta_{s-1} = \nu_s\beta^{-1}$.

$$\begin{array}{cccccccl}
z_2 & z_3 & \cdots & z_{s-1} & z_s & a_1a_1 & \cdots & a_1a_p \\
 & & & & & & & \downarrow \tau'' \\
a_1 & w_2 & \cdots & w_{s-2} & w_{s-1} & w_s & \cdots & w_r \\
 & & & & & & & \downarrow \eta_2\cdots\eta_{s-1} \\
w_2 & w_3 & \cdots & w_{s-1} & a_1 & w_s & \cdots & w_r \\
 & & & & & & & \downarrow \beta \\
z_2 & z_3 & \cdots & z_{s-1} & a_1 & z_sa_1 & \cdots & z_sa_p \\
 & & & & & & & \downarrow \nu_s^{-1} \\
z_2 & z_3 & \cdots & z_{s-1} & z_s & a_1a_1 & \cdots & a_1a_p
\end{array}$$

Computing the relation $\epsilon''\nu_s\cdots\nu_2\epsilon_2 = \epsilon$.

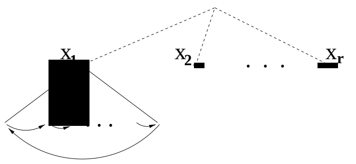
$$\left.\begin{array}{ccccccccl}
a_1 & w_2 & \cdots & w_{s-2} & w_{s-1} & w_s & \cdots & w_r & \\
 & & & & & & & & \downarrow \epsilon'' \\
z_2 & z_3 & \cdots & z_{s-1} & z_s & a_1(a_1\pi) & \cdots & a_1(a_p\pi) & \\
 & & & & & & & & \downarrow \nu_s \\
z_2 & z_3 & \cdots & z_{s-1} & a_1 & z_s(a_1\pi) & \cdots & z_s(a_p\pi) & \\
 & & & & & & & & \downarrow \nu_{s-1}\cdots\nu_2 \\
a_1 & z_2 & \cdots & z_{s-2} & z_{s-1} & z_s(a_1\pi) & \cdots & z_s(a_p\pi) & \\
 & & & & & & & & \downarrow \epsilon' \\
v_1 & v_2 & \cdots & v_{s-2} & v_{s-1} & v_s(a_1\pi) & \cdots & v_s(a_p\pi) &
\end{array}\right)\epsilon$$

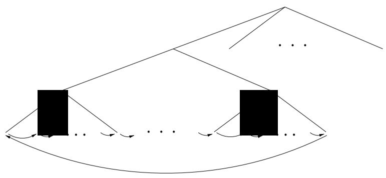
Computing the relation $\tau'\beta^{-1} = \tau$.

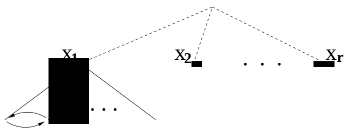
$$\left.\begin{array}{cccccccc}
u_1 & u_2 & \cdots & u_{s-1} & u_s a_1 & \cdots & u_s a_p & \\
 & & & & & & & \downarrow \tau' \\
a_1 & z_2 & \cdots & z_{s-1} & z_s a_1 & \cdots & z_s a_p & \\
 & & & & & & & \downarrow \beta^{-1} \\
a_1 & w_2 & \cdots & w_{s-1} & w_s & \cdots & w_r &
\end{array}\right) \tau$$

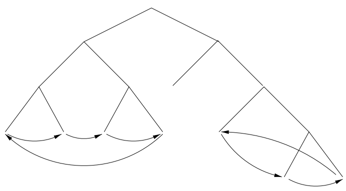
Appendix B

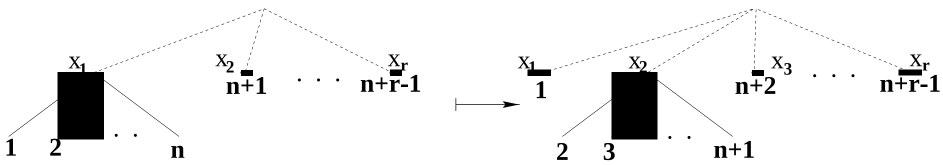
Tree Diagrams of Generators

$a =$ 

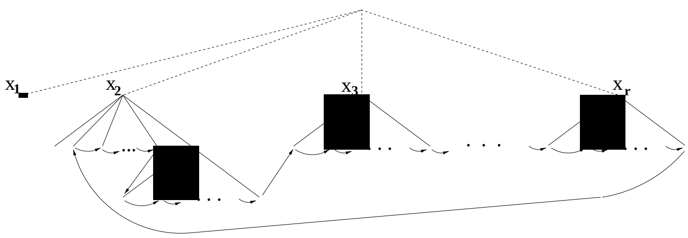
$\bar{a} =$ 

$\hat{a} =$ 

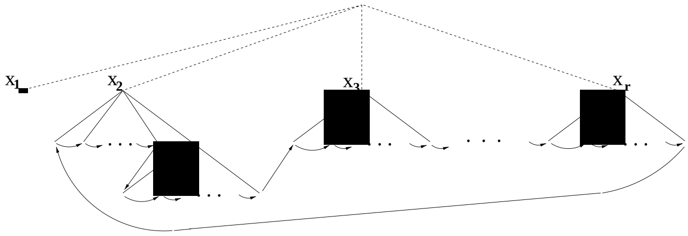
$d =$ 

$c =$ 

$b =$



, r and n odd

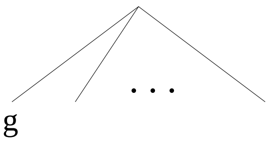


, otherwise

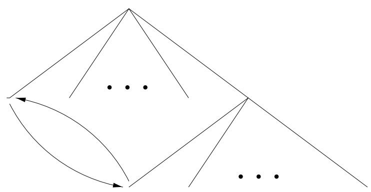
Appendix C

Tree Diagrams to Look at

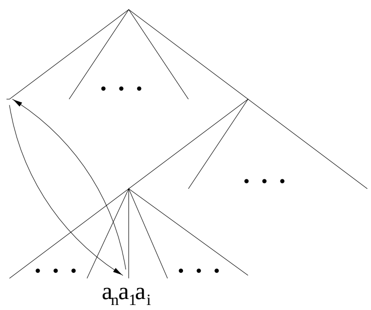
σg =

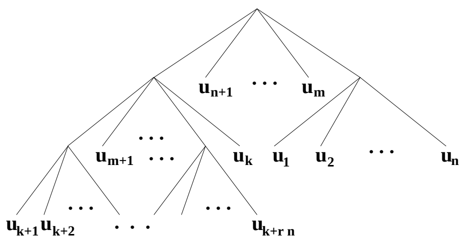


δ =

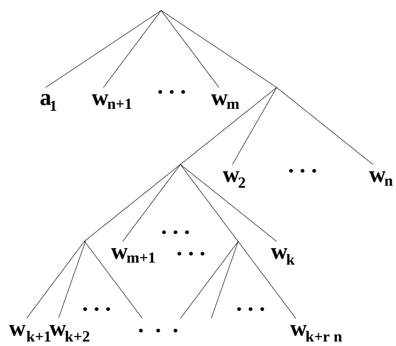


δ_i =





$m = 2n - 2$ and $k = m + t$



The top row of Γ and the basis corresponding to S.

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