Assignment 1

1. Find the simultaneous solutions of the two equations below **graphically** and then verify your answer.

$$\begin{cases} 2x & -y & = & -1 \\ -3x & -3y & = & 6 \end{cases}$$

2. Find all the solutions of the following systems of equations using Gaussian elimination.

(a) $$\begin{cases} x & +2y & & = & -1 \\ 2x & +4y & +5z & = & 8 \\ -x & +3y & -2z & = & -8 \end{cases}$$

(b) $$\begin{cases} 2x & -y & +3z & = & 4 \\ 9x & +2y & -z & = & 13 \\ -3x & -5y & +8z & = & -5 \end{cases}$$

Assignment 2

1. Evaluate the following products of matrices.

(a)
$$\begin{pmatrix} 2 & 7 & -3 \\ 11 & -2 & 0 \\ 3 & -1 & 5 \end{pmatrix} \begin{pmatrix} 6 & 3 \\ 2 & -3 \\ 8 & 4 \end{pmatrix}$$

(b)
$$\begin{pmatrix} 4 & -3 \\ 2 & 9 \\ 12 & 0 \end{pmatrix} \begin{pmatrix} -3 & 3 \\ 0 & 6 \\ -2 & -1 \end{pmatrix}$$

2. (a) For each of the conditions below find two 2×2-matrices A and B satisfying it.
   i. $AB \neq BA$
   ii. $AB = 0$ but neither $A = 0$ nor $B = 0$
   iii. $AB = C$, where C is obtained from B by interchanging its two rows and the two rows of B are not equal.

   (b) Can you do the same as in part (a) for 1×1-matrices? Justify your answer.

3. ;-) By arguing about 2×2-matrices of the form

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

show the addition theorems for sin and cos:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Assignment 3

1. Solve the following system of linear equations using **matrix notation**.

$$\begin{cases} 2x & -y & -z & = a \\ -x & +2y & -z & = b \\ -2x & y & +z & = c \end{cases}$$

where (a, b, c) is (i) $(3, 2, 1)$ and (ii) $(0, 0, -3)$.

2. (a) Find **all** real 2×2-matrices A satisfying $AA = -I_2$.

   (b) Use part (a) to show that

$$f(z) = \begin{pmatrix} \mathrm{Re}(z) & -\mathrm{Im}(z) \\ \mathrm{Im}(z) & \mathrm{Re}(z) \end{pmatrix}$$

   is a map from the complex numbers into 2-square matrices which respects multiplication and addition, i.e. for all complex numbers z_1 and z_2, $f(z_1 z_2) = f(z_1)f(z_2)$ and $f(z_1 + z_2) = f(z_1) + f(z_2)$.

3. ;-) For arbitrary but fixed $n \geq 2$ show that an n-square matrix D commutes with every other n-square matrix if and only if D is a scalar multiple of the identity matrix I_n.

Assignment 4

1. (a) Prove the following analogue of Theorem 7: When I is a matrix such that $IA = A$ for all matrices A such that the product IA exists, then $I = I_n$ for some n.

   (b) Let A and B be invertible n-square matrices. Show that AB is also invertible and find its inverse.

2.

3. ;-) A matrix A is called *lower triangular* if it is a square matrix and $A_{ij} = 0$ whenever $j > i$.

   (a) Prove that the inverse of an invertible lower triangular matrix is also a lower triangular matrix.

   (b) Find necessary and sufficient conditions for a lower triangular matrix to be invertible and justify your answer.

   (c) Let A be a lower triangular n-square matrix. Find an invertible matrix P such that $P^{-1}AP$ is an upper triangular matrix. Here a matrix U is called *upper triangular* if it is a square matrix and $U_{ij} = 0$ for $i > j$.

Assignment 5

1. Decide which of the following sequences of vectors are linearly independent and justify your answer.

(i) $\left(\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix} \right)$ (ii) $\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right)$

(iii) $\left(\begin{pmatrix} 4 \\ 8 \\ -2 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix} \right)$ (iv) $\left(\begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 13 \\ 0 \\ 13 \end{pmatrix} \right)$

(v) $\left(\begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 10 \end{pmatrix} \right)$ (vi) $\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 6 \end{pmatrix} \right)$

2. Invert the following matrices if possible.

(a) $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 0 & 11 \\ -4 & 2 & 0 \\ 0 & 2 & 11 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 2 & 1 & -1 \\ 2 & 2 & 1 & 1 \\ 0 & 8 & -2 & 1 \\ 3 & 3 & -1 & 0 \end{pmatrix}$

3. :-) A matrix A is called *nilpotent* if A is an n-square matrix satisfying $A^m = 0_n$ for some integer $m \geq 1$.
   (a) Let A be a nilpotent matrix with $A^m = 0$. Show that $A - I_n$ and $I_n + A + A^2 + \cdots + A^{m-1}$ are invertible.
   (b) Describe the set of n-square matrices A for which $A_{ij} = 0$ whenever $1 \leq j \leq i \leq n$ in words and prove that such matrices are nilpotent.
   (c) Use (a) and (b) to show that an upper triangular matrix whose diagonal entries are all non-zero is invertible. (See Question 3 on problem sheet four for the definition of an upper triangular matrix.)

Assignment 6

1. Let $\underline{v}_1 = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -1 \\ 5 \\ -1 \end{pmatrix}$ and $\underline{v}_2 = \begin{pmatrix} 9 \\ -6 \\ -3 \end{pmatrix}$.
   (a) Is the sequence of vectors $\langle \underline{v}_1, \underline{v}_2, \underline{v}_3 \rangle$ linearly independent?
   (b) For each of the following vectors decide whether it is a linear combination of $\underline{v}_1$, $\underline{v}_2$, and $\underline{v}_3$.

   (i) $\begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 6 \\ 5 \\ -2 \end{pmatrix}$ (iii) $\begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$

(c) Invert, if possible, the matrix whose columns are $\underline{v}_1$, $\underline{v}_2$, and $\underline{v}_3$.

2. Calculate the determinants of the following matrices.

$$(a)\quad \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix} \qquad (b)\quad \begin{pmatrix} -5 & 3 \\ -2 & 1 \end{pmatrix}$$

$$(c)\quad \begin{pmatrix} 1 & 2 & 0 \\ -1 & 2 & 1 \\ 0 & 3 & 6 \end{pmatrix} \qquad (d)\quad \begin{pmatrix} 7 & 2 & 2 \\ 3 & 1 & 1 \\ -3 & 4 & -3 \end{pmatrix}$$

3. ;-) The *transpose* of a matrix A is the matrix whose i-th row is the i-th column of A and it is denoted A^t. This is equivalent to the requirement that $(A^t)_{ij} = A_{ji}$ for all i and j which make sense.

   (a) Let A and B be matrices such that AB is defined. Show that the transpose of AB is B^tA^t.

   (b) Show that a square matrix A is invertible if and only A^t is invertible.

   (c) A matrix S is *symmetric* if $S = S^t$. Show that AA^t is symmetric whenever A is a square matrix.

Assignment 7

1. For 2×2-matrices A and B verify that $|AB| = |A||B|$.

2. Let α be real parameter. For each of the following matrices find the values of α for which the matrix is not invertible, by looking at their determinants.

$$(i)\quad \begin{pmatrix} 2+\alpha & 4 \\ -1 & \alpha \end{pmatrix} \qquad (ii)\quad \begin{pmatrix} 4 & \alpha^2 \\ -2 & -\alpha \end{pmatrix} \qquad (iii)\begin{pmatrix} 1 & \alpha & 0 \\ \alpha & 2 & \alpha \\ 0 & \alpha & 1 \end{pmatrix}$$

3. Show that 0 and 3 are eigenvalues of the matrix $\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$, by finding eigenvectors. Next, let P be the matrix whose columns are the eigenvectors of A. Find P^{-1} and evaluate $P^{-1}AP$.

4. :-)

   (a) Do Question 1) above.

   (b) Let a, b, and c be positive integers satisfying $a^2 + b^2 = c^2$, and put

$$A = \frac{1}{c}\begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

   Calculate $|A|$ and find A^{-1}.

(c) Now let $A_0 = A$ and, for $n \geq 1$, define

$$A_n = A_{n-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A_{n-1}.$$

Show that A_n is of the form $\frac{1}{c_n} \begin{pmatrix} a_n & b_n \\ b_n & -a_n \end{pmatrix}$ with a_n, b_n, and c_n integers. Use part (a) to show that $a_n^2 + b_n^2 = c_n^2$.[1]

(d) Find two linearly independent eigenvectors of A_1, when $a = 3$, $b = 4$, and $c = 5$.[2]

Assignment 8

1. Let $T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix}$. Find T^{-1} using elementary row operations and then compute $A = T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix} T^{-1}$.

2. Find the characteristic polynomial, the eigenvalues, and the eigenvectors of the following matrices.

$$(i) \quad \begin{pmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix} \qquad (ii) \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -2 \\ -2 & -2 & -1 \end{pmatrix}$$

3. :-) The *trace* of the n-square matrix A is, by definition, the sum of the diagonal entries. Writing $tr(A)$ for the trace of A we thus have $tr(A) = \sum_{i=1}^{n} (A)_{ii}$. Show that for every invertible n-square matrix T and arbitrary n-square matrix A, $tr(A) = tr(TAT^{-1})$. Hint: find the trace in the characterisitc polynomial.

Assignment 9

1. (a) Find the eigenvalues and corresponding eigenvectors of $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Find a reason why there does not exist a 2×2-matrix T such that $T^{-1}AT$ is a diagonal matrix.

   (b) Find the eigenvectors of $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and interpret your answer.

[1] I don't claim that this is the best way to prove that there are infinitely many integer solutions for the equation $x^2 + y^2 = z^2$

[2] Maybe you should read Question 3.

2. Give an example of a 3×3-matrix with eigenvectors

$$\underline{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \underline{v}_3 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

   with respective eigenvalues 2, 1, and -1.

3. :-) Take for granted that $\{1, x, x^2, x^3\}$ is a basis for the real vector space of all polynomials of degree at most three and with real coefficients.

   (a) Which column vector represents the polynomial $p(x) = 2 - x + 3x^2$?

   (b) Which linear mapping is described by the matrix $D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ with respect to the basis $\mathcal{B}$?

   (c) What are the eigenvectors of D?

   (d) Find a matrix which describes the same linear mapping as in (b) but with respect to the basis $\{1 - x, x + x^2, x^3, -2x^2\}$. Justify your answer.