2S1 Sample Scholarship Exam

January 23rd, 2004, Lecturer: Claas Röver

1. Find the area of the region R which is bounded by the following four curves
$$y = x, \quad y = \frac{1}{3}x, \quad y = \frac{1}{x} \quad \text{and} \quad y = \frac{2}{x}.$$

2. State the first and second partial derivative tests for relative maxima, relative minima and saddle points of a function $f(x, y)$.
A company is asked to build a closed rectangular box whose volume is 72m^3 using two kinds of material. The material to be used for the top and bottom costs 27 euros per square meter and the sides cost 9 euros per square meter. Find the dimensions and the price of the cheapest such box.

3. Let f be a function of the variables x and y with continuous second partial derivatives. If $x = u^2 + v^2$ and $y = cuv$, where c is a nonzero constant, show that
$$\frac{\partial^2 f}{\partial u \partial v} = c\left(\frac{4}{c^2} y \frac{\partial^2 f}{\partial^2 x} + y \frac{\partial^2 f}{\partial^2 y} + 2x \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial y}\right).$$

4. State a formula expressing the relationship between the directional derivative of f in the direction of a unit vector u and the gradient of f. What does this mean for points at which the gradient is zero?
Find the directional derivatives of $f(x, y) = \frac{9}{x^2+y^2}$ at the points $P = (1, 1)$, $Q = (1, -1)$ and $R = (0, \sqrt{2})$, each time in the direction towards the origin. Explain, why you got the same answer in all three cases.

5. Evaluate
$$\int_0^4 \int_{\sqrt{y}}^2 e^{x^3/8}\, dx\, dy$$
by first changing the order of integration.

6. Find the volume of the solid which is enclosed between the cone $f(x, y) = \sqrt{x^2 + y^2}$ and the paraboloid $g(x, y) = 6 - x^2 - y^2$.

7. A satellite in the shape of an ellipsoid $4x^2 + y^2 + 4z^2 = 9$ has gone out of control crashes back to earth. As it enters the atmosphere it heats up, and the temperature T on its surface is found to be $T = x^2 y$. Find the hottest points on the satellite's surface using the method of Lagrange multipliers.