2S1 Sample Scholarship Exam

February 16th, 2004, Lecturer: Claas Röver

In the real exam you only need to do seven questions!

1. Define what is meant by the phrase "$\overrightarrow{F}(x, y, z)$ is a conservative vector field in the region D".
   Show that the vector field $\overrightarrow{F}(x, y, z) = (3y + z, 3x - 2z^2, x - 4yz)$ is conservative everywhere in $\mathbb{R}^3$ and compute the line integral
   $$\int_C \overrightarrow{F} \cdot d\overrightarrow{r},$$
   where C is the parametric curve given by $x(t) = t$, $y(t) = e^t$ and $z(t) = \sin(\pi t)$ with $0 \le t \le 1$.

2. State the first and second partial derivative tests for relative maxima, relative minima and saddle points of a function $f(x, y)$ of two independent variables. Define a critical point.
   Find all local maxima, local minima and saddle points of the function
   $$f(x, y) = -\frac{1}{4}x^4 + \frac{2}{3}x^3 + 4xy - y^2.$$

3. State the formula for the surface area of a parametric surface. Then determine the constant u- and constant v-curves of the parametric surface given by $x(u, v) = u + v$, $y(u, v) = 2v$ and $z(u, v) = \cos u$, and compute the area of the surface when $0 \le u \le \frac{\pi}{2}$ and $0 \le v \le \sin u \cos u$.

4. Evaluate the double integral
   $$\iint_R \frac{x}{2y}\, dA,$$
   where R is the region bounded by the four curves $y = 4 - x^2$, $y = 2 - \frac{1}{2}x^2$, $y = x^2$ and $y = 3x^2$.

5. Let S be the surface given by $z = f(x, y) = \sqrt{x^2 + 9y^2}$ and let P be the point $(2, 1, 0)$. Determine the point Q on the surface S which is closest to P and give an equation for the plane which goes through P, Q and the origin.

6. Let $\mathbf{F}$ be a three dimensional vector field whose component functions have continuous second partial derivatives. Prove that the divergence of the curl of $\mathbf{F}$ is zero.

7. Evaluate the line integral

$$\int_C e^x \sin y \, dx + e^x \cos y \, dy,$$

where C is the closed curve consisting of the semicircle $y = \sqrt{1 - x^2}$ and the interval $[-1, 1]$.

8. Fix $a \geq 0$ and let C be the curve defined by $x^3 + y^3 = 3axy$. Show that C can be parametrised by putting $x(t) = \frac{3at}{1+t^3}$ and $y(t) = \frac{3at^2}{1+t^3}$ where $t \neq -1$. Then argue that C forms a loop in the first quadrant ($x \geq 0$, $y \geq 0$) and compute the area enclosed by this loop.