3E1 Problem Sheet 11*

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1. Find the image of the following regions under the mapping $f(z) = e^z$.

   (a) The strip of all complex numbers z with $-\pi < \mathrm{Im}(z) < 0$ and $\mathrm{Re}(z) \geq 0$.

   (b) The square with vertices 0, $\frac{i\pi}{2}$, $\frac{\pi}{2}$ and $\frac{\pi}{2}(1+i)$.

   Give a mathematical description, as well as a sketch of the regions and their images.

2. State the definitions of the complex sine and cosine functions.
   Recall that $\cosh y = \frac{1}{2}(e^y + e^{-y})$ and $\sinh y = \frac{1}{2}(e^y - e^{-y})$ for $y \in \mathbb{R}$.

   Now verify the following identities where $z = x + iy$.

   (a) $\sin z = \sin x \cosh y + i \cos x \sinh y$.

   (b) $|\sin z|^2 = \sin^2 x + \sinh^2 y$. You may use $1 = \cosh^2 y - \sinh^2 y$.

   Say in words what the formula in (a) expresses.

3. State the Cauchy-Riemann equations for an analytic function $f(z) = g(x,y) + ih(x,y)$, where $z = x + iy$ and g and h are real functions of two variables.
   Now let $f(z)$ be an analytic function such that $\mathrm{Re}(f(z)) = c = \mathit{const}$. Prove that $f(z) = C = \mathit{const}$. ($C$ may be a complex number.)

*Remember that Question 1 is not neccessarily the easiest!