3E1 Problem Sheet 7

November 24 - 30, 2003

Lecturer: Claas Röver

1. Recall from the lectures that the method of separation of variables, i.e. setting $u(x,t) = F(x)G(t)$, applied to the one-dimensional heat (or diffusion) equation
$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{(HE)}$$
lead to the following two ordinary differential equations
$$\frac{\partial^2 F}{\partial x^2} - kF = 0 \quad \text{and} \quad \frac{\partial G}{\partial t} - kc^2 G = 0,$$
where k is the so called *separation constant.*

   (a) Let L denote the length of the wire under consideration, so $L > 0$. Show that the assumption $k \geq 0$ together with the boundary conditions
$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(L,t) = 0 \qquad \text{(BC}'\text{)}$$
imply that $u(x,t) = K = const.$

   (b) Why do you need to consider the cases $k = 0$ and $k > 0$ separately in part (a)?

   (c) Say in words what the boundary conditions (BC′) mean.

2. From the lecture, the general solution of (HE) satisfying (BC′) is
$$u(x,t) = \sum_{n=0}^{\infty} A_n \left(\cos \frac{n\pi x}{L}\right) e^{-\lambda_n^2 t}\,, \qquad \text{where } \lambda_n = \frac{cn\pi}{L},\; n = 0, 1, 2, \ldots$$
(Here $c^2 = \frac{K}{\sigma\rho}$ is a constant depending on the material of the wire.) Find a solution of (HE) satisfying (BC′) and the initial condition
$$u(x,0) = 8x(L-x) \quad \text{for} \quad 0 \leq x \leq L. \qquad \text{(IC)}$$

3. In Question 2 on Sheet 6 you should have found that
$$v(x,t) = \frac{64\,L^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \left(\sin \frac{(2n-1)\pi x}{L}\right) e^{-\lambda_n^2 t}$$
is the solution of (HE) satisfying (IC) and the boundary conditions
$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0. \qquad \text{(BC)}$$
Compare this with your answer to Question 2 above for very large values of t and explain the result.