3E1 Problem Sheet 8

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1. Recall that the eigenfunctions of a rectangular membrane of side lengths a and b are
$$u_{mn}(x, y, t) = (a_{mn} \cos \lambda_{mn} t + b_{mn} \sin \lambda_{mn} t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b},$$
for positive integers m and n and with $\lambda_{mn} = c\pi \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$ and $c^2 = \frac{T}{\rho}$.
   (a) Assume that $a = b = 1$. Find all eigenfunctions with eigenvalue $c\pi\sqrt{85}$.
   (b) Assume that $a = 1$ and $b = 2$. Find two eigenfunctions with the same eigenvalue.
   (c) In the case where a and b are arbitrary, state what the frequency of the eigenfunction u_{mn} is and hence find the eigenfunction with the lowest frequency?

2. In order to satisfy the initial condition $u(x, y, 0) = f(x, y)$ we took a superposition of all the eigenfunctions and found that we have to find coefficients a_{mn} such that
$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = f(x, y).$$
By setting $K_m(y) = \sum_{n=1}^{\infty} a_{mn} \sin \frac{n\pi y}{b}$ and not worrying about convergence, derive the generalized Euler formula
$$a_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy.$$

3. Find the solution of the two-dimensional wave equation for a rectangular membrane of side lengths a and b with the following initial conditions
   (a) $u(x, y, 0) = 4 \sin \frac{3\pi x}{a} \sin \frac{2\pi y}{b}$ and $\frac{\partial u}{\partial t}(x, y, 0) = 0$.
   (b) $u(x, y, 0) = x(a - x) \sin \frac{\pi y}{b}$ and $\frac{\partial u}{\partial t}(x, y, 0) = 0$.

   Hint: In part (a) you don't need a complicated computation! For part (b) recall the orthogonality of the sines.